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Critical and off-critical singularities in disordered quantum magnets

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Abstract. We study disordered systems at and near quantum phase transitions, which are transitions at zero temperature that are driven by quantum instead of thermal fluctuations. Special emphasis is put on the so-called Griffiths-McCoy regions surrounding these transitions in magnetic systems, where various susceptibilities are found to be divergent for $T \to 0$ without the system being critical. We present a phenomenological picture for uncorrelated and weakly or strongly correlated disorder and check it for various models. In higher dimensions the phenomena seem to be connected to a percolation transition.

Keywords: Quantum phase transitions; Griffiths-McCoy region; Quantum magnets; Bose glass phase

1 Introduction

Quantum phase transitions are phase transitions at zero temperature driven by quantum fluctuations rather than thermal fluctuations. If disorder is involved, as in random magnets, in amorphous bosonic systems or at the Mott-Anderson transition, one encounters new features that are usually absent in pure systems. In particular at a quantum critical point one observes new universality classes with an strongly anisotropic scaling of space and (imaginary) time. Moreover one finds regions in the phase diagram, where various susceptibilities diverge without the system being critical, commonly referred to as quantum Griffiths-McCoy phases [1, 2, 3]. The latter is most prominent in the context of random transverse Ising models, being realized by e.g. the dipolar Ising system $LiHo_xY_{1-x}F_4$ [4], however also occur as Bose glass phase of localized bosons in amorphous superconducting films [5], Helium-4 in or on aerogels [6], or disordered Josephson junction arrays. Moreover, it has been argued that experimentally observed non-Fermi liquid behavior in f-electron compounds featuring continuously varying exponents are a manifestation of a Griffiths phase [8].
These quantum Griffiths singularities can be parameterized by a single dynamical exponent $z(\delta)$, varying continuously with the distance δ from the critical point. One of the most intriguing questions is whether, although describing different physics, $z(\delta)$ merges with the critical dynamical exponent $z_{\rm crit}$ when approaching the critical point $\delta \to 0$. Another is, whether typical and averaged quantities scale differently at the critical point. In one space dimension considerable progress has been made in

understanding these issues [9, 10, 11]. However, in higher dimensions one has to rely on quantum Monte-Carlo methods and progress can only be made by a significant computational effort [13, 15, 16].

2 The random transverse Ising model and its critical point

In this paper we describe the phenomenology of the Griffiths-McCoy phase in a general framework including the effects of correlated disorder in disordered transverse Ising models. The reason for considering correlated disorder, besides academic interest, is that it has been argued [8] that the non-fermi liquid behavior observed in rare earth alloys might be the manifestation of the Griffiths-McCoy singularities occurring in an effective model, a transverse field spin-1/2 model Ising model with correlated disorder. To be specific we consider here model for a disordered quantum magnet with a strong Ising anisotropy and transverse fields:

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij}\sigma_i^z\sigma_j^z + \sum_i \Gamma_i \sigma_i^x \tag{1}$$

in which σ_i are spin-$\frac{1}{2}$ operators located on a d-dimensional lattice, $\langle ij \rangle$ indicate all nearest neighbor pairs on this particular lattice, $J_{ij} \geq 0$ *ferromagnetic* interactions (a mixture of ferro- and antiferromagnetic interactions causes frustration and leads to a quantum spin glass, which has been studied in [13, 3]) and Γ_i transverse fields that are randomly distributed modeling a quenched (i.e. time-independent) disorder.
For uncorrelated disorder one takes all random variables to be independently distributed. Here we discriminate different types of distribution:

pure case	$\Gamma_i = \Gamma$, $J_{ij} = J$: **no disorder**
bond-dilution	$\Gamma_i = \Gamma$, $J_{ij} = J$ w. prob. p, $J_{ij} = 0$ w. prob. $1-p$,
site-dilution	$\Gamma_i = \varepsilon_i\Gamma$, $J_{ij} = \varepsilon_i\varepsilon_j J$, $\varepsilon_i = 1$ w. prob. p, $\varepsilon_i = 0$ w. prob. $1-p$
binary	$\Gamma_i = \Gamma$, $J_{ij} = J_1$ or J_2, $(J_1 < J_2)$ w. prob. $1/2$
uniform	$\Gamma_i \in [0,\Gamma]$, $J_{ij} \in [0,J]$, both uniformly distributed

In one dimension bond or site dilution leads to a fragmentation of the system into finite, homogeneous chain segments. This is a nice toy model, which does not, however have a phase transition and therefore we consider dilution only for $d \geq 2$. In general, for p above the percolation threshold $p > p_c$ in the diluted case ($p = 1$ in $d = 1$), the quantum system described by eq. (1) has a second order phase transition at zero temperature that manifests itself in specific macroscopic properties of the ground state and low lying excitations. A critical value Γ_c for the transverse field strength separates a disordered or paramagnetic phase for $\Gamma > \Gamma_c$ from an ordered phase for $\Gamma < \Gamma_c$. In $1d$ this point is exactly know due to the fact that the model is self-dual here, the critical point being simply given implicitly by the equation $[\ln J_{i,i+1}]_{\rm av} = [\ln \Gamma_i]_{\rm av}$, where $[\ldots]_{\rm av}$ indicated the average over the distribution of the disorder distributions, i.e. the bonds and fields (note that for the pure case this yields $\Gamma_c = J$). In higher dimensions the critical point is generally not known and has to be determined numerically. The transition is characterized by a diverging length scale $\xi \sim |\Gamma - \Gamma_c|^{-\nu}$ and a vanishing characteristic frequency $\omega \sim \Delta E \sim \xi^{-z}$. The latter is the quantum

analog of "critical slowing down" in the critical dynamics of classical, thermally driven transitions. Together with a third critical exponent, defining the anomalous dimension of the order parameter field (the magnetization), the thermal exponent ν and the dynamical exponent z give a complete description of the transition via a set of scaling relations.

In $1d$ it is by now well established [9, 10, 11] that various physical quantities have a logarithmically broad distribution, which forces one to discriminate between average and typical properties. For instance the typical correlation length diverges with an exponent $\nu_{\mathrm{typ}} = 1$, whereas the average diverges with $\nu_{\mathrm{av}} = 2$. It turns out that it is not the energy gap but its logarithm that scales with system size and distance from the critical point, giving rise to an exponential rather than algebraic decrease of the energy gap with system size: $\Delta E \sim \exp(-aL^{1/2})$. This means that $z = \infty$ and since the inverse gap corresponds to a characteristic relaxation or tunneling time it is reminiscent of an activated dynamics scenario in conventional spin glasses or random field systems. Moreover, away from the critical point the time dependent correlations still decay algebraically although the spatial correlation length is finite. In this so-called Griffiths-McCoy region [1, 2] near the QPT various susceptibilities are still divergent depending on a particular parameter $z(\delta)$, the dynamical exponent in the Griffiths-McCoy phase, which characterizes the strengths of the singularities encountered in a distance δ away from the QPT [3].

Again, in $1d$ this scenario, QPT and Griffiths-McCoy region, has been worked out in much detail [9, 10]. Various powerful analytical and numerical tools, which are specific for one space dimension, facilitate the thorough study of this paradigmatic system. To obtain an understanding for the underlying physics responsible for such an unusual behavior and to get an intuitive idea of what to expect in higher dimension it is useful to have the phenomenological picture in mind that we are going to present now.

3 The Griffiths-McCoy phase: Off-critical singularities

The origin of the Griffiths-McCoy singularities emerging in the paramagnetic (and also in the ferromagnetic) phase well away from the critical point are strongly coupled clusters (SCC) of arbitrary size (which occur with a probability that decreases monotonically, and often very fast, with increasing cluster size. What a SCC is becomes most evident in the diluted case: here they are simply the connected compact clusters occurring for concentrations below the percolation threshold $p < p_c$. For the binary distribution these are analogously clusters that contain dominantly spins interacting via the strong coupling constant J_2. For a continuous distribution like the uniform case it is more difficult to define a SCC, however, it is straightforward in a more general framework [9, 10].

Thus one ingredient for estimating the anomalous behavior of the system (anomalous with respect to the normal behavior of the pure system in the off-critical region) is the probability $P_{\mathrm{SCC}}(V)$ with which such a SCC of volume V occurs (more precisely it is the probability with which a spin belongs to s SCC of volume V). For uncorrelated disorder this is simply exponentially small in the volume (e.g. for the diluted case it is $P_{\mathrm{SCC}}(V) = p^V$, or $P_{\mathrm{SCC}}(V) = \exp(-cV)$, with c being a positive constant. For correlated disorder, which might be important for the above mentioned effective

models for the non-fermi liquid behavior $P_{\rm SCC}(V)$ is enhanced:

uncorrelated disorder	$P_{\rm SCC}(V) \sim \exp(-cV)$
weakly correlated	$P_{\rm SCC}(V) \sim \exp(-cV^{\beta})$ with $\beta < 1$
strongly correlated	$P_{\rm SCC}(V) \sim (cV)^{-\alpha}$ with $\alpha > 1$

Obviously in a SCC the spins have the tendency to order (ferromagnetically) when one is close enough to the critical point. This local order manifests itself in a very long relaxation time, in the classical case [12] as well as in the quantum mechanical case we are considering here. An estimate for the zero temperature relaxation, or tunneling, time τ of a SCC of volume V is, via first order perturbation theory,

$$\tau(V) \sim \exp(\sigma V) \ , \tag{2}$$

where σ is a constant. It is the fact that $\tau(V)$ is exponentially large in the volume (and not the surface of the SCC as in the classical case [12]) which leads to drastic consequences, singularities and even divergences in physical observables in the off-critical region near a QPT.
The cluster distribution $P_{\rm SCC}(V)$ translates via (2) in a probability distribution for relaxation times (more precisely the probability for a spin to relax with a characteristic relaxation time τ):

$$P(\tau) = P_{\rm SCC}\big(V(\tau)\big) \cdot \frac{dV(\tau)}{d\tau} \sim \begin{cases} \tau^{-1-c/\sigma} & \text{for uncorr. dis.} \\ \tau^{-1}\exp\{-c/\sigma \cdot (\ln\tau)^{\beta}\} & \text{for } \textit{weakly} \text{ corr. dis.} \\ \tau^{-1}\{c/\sigma \cdot \ln(\tau)\}^{-\alpha} & \text{for } \textit{strongly} \text{ corr. dis.} \end{cases} \tag{3}$$

The constant constant c/σ, which depends on the distance δ from the critical point through the geometric constant c and/or the stiffness constant σ, is related to a *dynamical exponent* $z(\delta)$, since it occurs naturally in a dimensionless scaling combination of length and time scale: In a finite system of linear dimension L the probability for a spin to belong to a SCC of a particular volume V is proportional to the system size L^d, thus, for uncorrelated disorder, in a finite system

$$\tau P_L(\tau) \propto L^d \cdot \tau^{-c/\sigma} = (L\tau^{1/z(\delta)})^d \quad \text{with} \quad d/z(\delta) = c/\sigma \ . \tag{4}$$

As we will see, the dynamical exponent $z(\delta)$ parameterizes all off-critical singularities in of the Griffiths-McCoy phase.
To make contact with a physical observable one considers the spin-spin autocorrelation-function $C(t) = [\langle \sigma_i^z(t)\sigma_i^z(0)\rangle]_{\rm av}$, which is usually expressed in terms of the distribution of local relaxation times by

$$[C(t)]_{\rm av} = \int_{\tau_0}^{\infty} d\tau\, P(\tau)\, e^{-t/\tau} \quad \approx \ t^{-d/z(\delta)} \tag{5}$$

where τ_0 is a microscopic lower cut-off-time and the last equality is for uncorrelated disorder, which yields an algebraic decay. For weakly correlated one gets a decay that is slower than algebraic and for strongly correlated disorder a logarithmically slow decay, similar to the ultra-slow relaxation *at* the critical point in $1d$ [11, 14].

Also *static* quantities are affected by these singularities: for instance the local zero-frequency susceptibility

$$[\chi_{\rm loc}(\omega = 0)]_{\rm av} = \int_0^\beta dt\, [C(t)]_{\rm av} = \int_{\tau_0}^\infty d\tau\, (1 - e^{-\beta/\tau}) \cdot \tau P(\tau) \tag{6}$$

has an algebraic singularity at zero temperature for uncorrelated disorder $[\chi_{\rm loc}(\omega = 0)]_{\rm av} \sim T^{d/z(\delta)-1}$ with a continuously varying exponent, which is reminiscent of the non-Fermi liquid behavior observed in rare earth alloys [8]. Similarly the local non-linear susceptibility has an even stronger divergence, $[\chi_{\rm nl}(\omega = 0)]_{\rm av} \sim T^{d/z(\delta)-3}$, analogously for higher derivatives of the magnetization. The linear susceptibility *diverges* for $z(\delta) > d$, the nonlinear susceptibility diverges for $z(\delta) > d/3$, etc. Moreover, the specific heat has also a singular behavior $c(T) \sim T^{d/z(\delta)}$. For correlated disorder, these algebraic singularities become even stronger.

Before we present the results that support the above phenomenological picture, let us emphasis that the dynamical exponent $z(\delta)$ is *not* a critical exponent, at least not in the usual sense. As we have pointed out above, the spatial correlations are short ranged in the off-critical region $\delta \neq 0$, only correlations in (imaginary) time are, on average, long ranged. One might call the region, where this happens a *semi-critical* line, since the system is only "half" critical, namely in the (imaginary) time direction. One of the most important question here is, how do these off-critical singularities influence the critical behavior at $\delta = 0$? In particular, is there a relation between the dynamical critical exponent $z_{\rm crit}$ and the limit $\lim_{\delta\to 0} z(\delta)$? The Griffiths-McCoy singularities are completely induced by the disorder and its local, time-independent (quenched) fluctuations, whereas critical behavior is usually of totally different origin. In particular one might imagine that the collective critical behavior involving length scales overrules the disorder effects. To make the story short: in $1d$ disorder completely determines the critical behavior of the system (e.g. $z_{\rm crit} = \lim_{\delta\to 0} z(\delta)$), and there are strong indications that this holds in higher dimensions, too.

3.1 The one-dimensional case: Analytical and numerical results

Let us now proceed with the presentation of results confirming the above picture. First we consider the $1d$ case, in which various analytically exact predictions exist and which can serve us also as a test for the numerical methods for higher dimensions. In $1d$ a connection with Siani's model for anomalous diffusion in a disordered environment yields an exact result for the dynamical exponent in the Griffiths-McCoy phase [14]. It is given by the implicit equation

$$\left[\left(\frac{J}{h}\right)^{1/z(\delta)}\right]_{\rm av} = 1\ . \tag{7}$$

For any distribution of J and h one obtains immediately the result $1/z = 2\delta + \mathcal{O}(\delta^2)$, with $\delta = ([\ln h]_{\rm av} - [\ln J]_{\rm av})/({\rm var}[\ln {\rm h}] + {\rm var}[\ln {\rm J}]) \ll 1$ being an appropriately normalized distance from the critical point. For the uniform distribution the dynamical exponent $z(\delta)$ is given by the solution of the equation $z\log(1 - z^{-2}) = -\ln\Gamma \quad (= -2\delta)$, which we depict in Fig. 1.

On the other hand, $z(\delta)$ can be obtained via the analysis of various quantities in the system, as we have discussed above. The easiest way seems to be to determine the gap

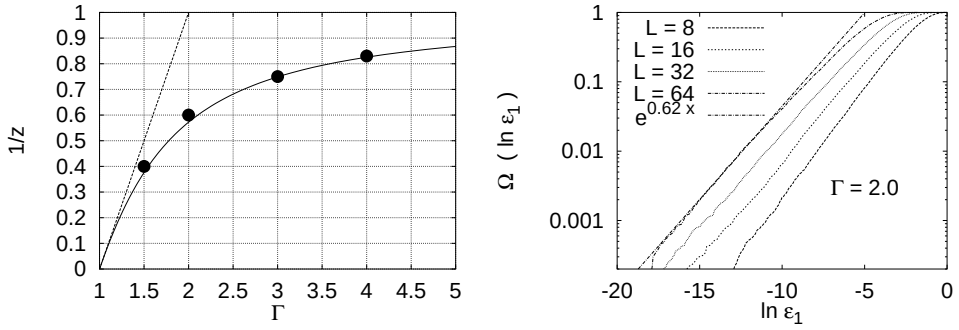

Fig. 1 Left: Exact result for the dynamical exponent $1/z(\delta)$ in $1d$ for the uniform distribution, see text ($\delta = 1/2 \ln \Gamma$). Note the asymptotic behavior at for approaching the critical point $\delta \to 0$, namely $1/z(\delta) = 2\delta + \mathcal{O}(\delta^2)$ and the asymptotic value $\lim_{\delta \to \infty} z(\delta) = 1$. The thick points are the numerical estimates obtained via the analysis of the probability distribution of the gap as it is exemplified in the right figure. Right: The integrated gap probability distribution $\Omega_L(\ln \epsilon_1)$ in the disordered phase ($\Gamma > 1$) for $\Gamma = 2$. The dynamical exponent $z(\delta)$ is extracted from the expected asymptotic form $\ln \Omega_L(\ln \epsilon_1) = 1/z(\delta) \ln \epsilon_1 + \text{const}$ which is a straight line when using a logarithmic scale on the y-axis. Thus $1/z(\Gamma = 2) \approx 0.62$. Note that for large Γ, i.e. far away from the critical point, there is essentially no system size dependence, since the spatial correlation length is expected to be small.

ϵ_1 (i.e. the energy difference between the ground state and the first excited state) for finite systems, since this is related to the longest relaxation time $\tau_{\max} \sim 1/\epsilon_1$. From sampling many (e.g. 10^5) disorder configuration one then generates the integrated probability distribution

$$\Omega_L(\ln \epsilon_1) = \int_{-\infty}^{\ln \epsilon_1} dy P_L(y) \; . \tag{8}$$

for the logarithm (which is more convenient) of the gaps. In Fig. 1 one can see that the integrated probability distribution for low energies is approximately a straight line on a log-log plot and from the slope one can estimate $1/z(\delta)$ quite accurately. A comparison with the exact data in Fig. 1 shows a good agreement.
By approaching the critical point the numerical estimate of $z(\delta)$ becomes difficult: now finite size effects become harder to control since the correlation length ξ diverges and an estimate of the slope of Ω_L is only possible if $L > \xi$. The fact that $\lim_{\delta \to \infty} z(\delta) = \infty$ implies that the distribution of gaps becomes logarithmically broad [10].

3.2 The diluted model: The percolation transition as a QPT

Next let us consider higher dimensions $d > 1$, and here we focus on uncorrelated disorder exclusively. We start with the diluted case, which is special, since here a percolation transition determines the critical and off-critical singularities. We focus on the region around the percolation threshold $p = p_c$ at small enough transverse fields Γ, where the transition line in a p-Γ phase diagram is vertical [16], see Fig. 2. Close to p_c the system consists of non-percolating clusters with a size distribution

$$P(V) \sim V^{-\tau} \exp\left(-\frac{aV}{\xi^D}\right) \quad \text{with} \quad \xi \sim |p - p_c|^{-\nu_p} \; , \tag{9}$$

where ν_p is the correlation length exponent and D is the fractal dimension of the percolating cluster. Hence, one obtains for the dynamical exponent $z(\delta)$

$$d/z = \begin{cases} -\ln(1-p)/\sigma & \text{for} \quad p \ll p_c \\ a/\sigma\xi^D & \text{for} \quad p \to p_c\,. \end{cases} \tag{10}$$

From (10) one concludes that z diverges algebraically at p_c [16]

$$z \propto |p - p_c|^{-D\nu_p} \quad \text{for} \quad p \to p_c \quad \text{and} \quad \Gamma \leq \Gamma_M. \tag{11}$$

This prediction can again be checked numerically, this time via quantum Monte-Carlo simulations [16] (using a continuous imaginary time cluster algorithm [15]) of the site-diluted transverse Ising model in $2d$. The gap, that we considered in the $1d$ case, remains inaccessible here, therefore we considered the integrated probability distribution of the local susceptibility $\chi_{\rm loc}(\omega = 0)$ given by (6), which is related to the inverse gap $\chi_{\rm loc}(\omega = 0) \sim 1/\epsilon_1$, i.e. proportional to the relaxation time τ [3, 10]. The analysis of the asymptotic slope of this distribution yields the numerical estimates for the dynamical exponent $z(\delta)$ and the agreement with the prediction (11) is good, as can be seen from Fig. 2.

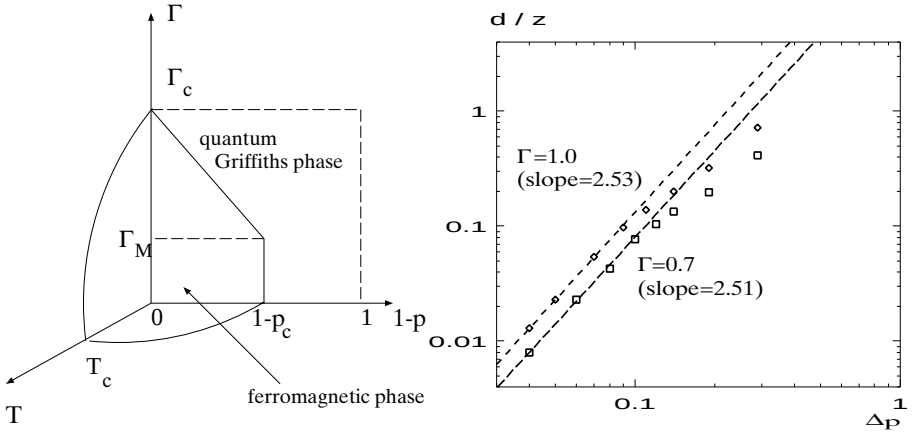

Fig. 2 Left: The phase diagram of the diluted transverse Ising ferromagnet. T_c is the transition temperature of the pure Ising ferromagnet (e.g. $T_c = 2.26$ in $2d$, $T_c = 4.51$ in $3d$), Γ_c is the critical transverse field of the pure transverse Ising model at T=0 (e.g. $\Gamma_c = 1$ in $1d$ in which case $T_c = 0$, $\Gamma_c = 3.04$ in $2d$), and p_c is the percolation threshold (e.g. $p_c = 0.5$ for bond dilution and $p_c = 0.41$ for site dilution in $2d$). Note the vertical line in the (p,Γ)–plane at p_c. Right: The estimate of the exponent d/z obtained by analyzing the probability distribution of the local susceptibility. The x-axis is $\Delta p = |p - p_c|$ with $p_c = 0.59$, the percolation threshold for site dilution in $2d$. The value of $D\nu_p$ obtained from the percolation theory is known to be $D\nu_p \simeq 2.57$. The agreement with the prediction (11) is good.

3.3 The random ferromagnet in $d > 1$

Thus for the diluted case in higher dimension $d > 1$ we encounter a situation that is very similar to $d = 1$. However, the underlying physics in the diluted case is a geometric phenomenon, the percolation transition, which is absent for other generic distributions like the uniform distribution, which we study next. Via extensive quantum

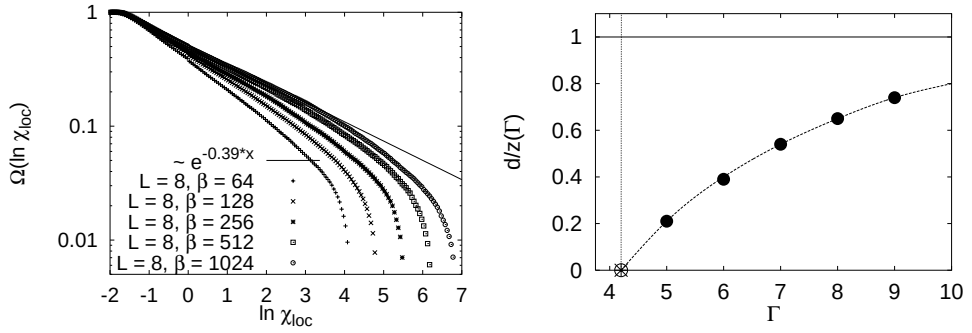

Fig. 3 Left: The integrated probability distribution of the local susceptibility $\chi_i(\ln\omega = 0)$ of the 2d random TIFM in the Griffiths-McCoy region at $\Gamma = 6.0$. The systems sizes L are larger than the spatial correlation lengths, which is small that far away from the critical point ($\Gamma > \Gamma_c \approx 4.2$). There is, however, a dependency on the system size in the imaginary time direction, which is β, the inverse temperature, since here the system has long ranged correlations. Therefore β has to be chosen large enough to read off the correct slope of the tail of the probability distribution. Right: The value of $d/z(\Gamma)$ obtained from analyzing the integrated probability distribution of $\chi_{\rm loc}(\ln\omega = 0)$. The vertical line indicates the (approximate) region of the critical point at $\Gamma_c \sim 4.2$, the open circle corresponds to $z(\Gamma_c) = \infty$ and the horizontal line at $d/z = 1$ indicates the expected limit $\lim_{\Gamma\to\infty} z(\Gamma) = d$.

Monte-Carlo simulations that utilize a continuous imaginary time cluster algorithm [15], which is very much in the spirit of other continuous time quantum Monte-Carlo algorithms [17, 18], the phase diagram of this system has been estimated recently [15]. The QPT of this system turns out to be located at $\Gamma_c \approx 4.2$ [15], so that for Γ larger than this value spatial correlations should be short ranged. However, due to the presence of strongly coupled regions in the system the probability distribution of excitation energies (essentially inverse tunneling times for these ferromagnetically ordered clusters) becomes extremely broad. Consequently we expect the probability distribution of local susceptibilities to have an algebraic tail at $T = 0$ [10, 3]

$$\Omega(\ln\chi_{\rm loc}) \approx -\frac{d}{z(\Gamma)}\ln\chi_{\rm loc} \tag{12}$$

where $\Omega(\ln\chi_{\rm loc})$ is the probability for the logarithm of the local susceptibility χ_i at site i to be larger than $\ln\chi_{\rm loc}$. The dynamical exponent $z(\Gamma)$ varies continuously with the distance from the critical point and parameterizes the strengths of the Griffiths-McCoy singularities also present in other observables. At finite temperatures Ω is chopped off at β, and close to the critical point one expects finite size corrections as long as L or β are smaller than the spatial correlation length or imaginary correlation time, respectively. We used $\beta \leq 1000$ and averaged over at least 512 samples. In Fig. 3 we show as an example data for $\Omega(\ln\chi_{\rm loc})$ at $\Gamma = 6.0$. From the asymptotic slopes 0.39 one derives via (12) the estimates $z(\Gamma = 6) = 5.1$. Collecting the estimates for $z(\Gamma)$ for other values of Γ one gets strong indications for a divergence $\lim_{\Gamma\to\Gamma_c} z(\Gamma) = \infty$, as already observed in the one-dimensional case. This concurs with investigations *at* the critical point, where $z_{\rm crit}$ is also found to be infinite [21].

One is confronted with the intriguing question, why the one-dimensional and the two-dimensional random bond ferromagnet in a random transverse field behave very simi-

lar, *at* the quantum critical point as well as in the Griffiths-McCoy phase. One might speculate, from the viewpoint of the real space renormalization group treatment carried through for the one-dimensional case [9], that the disorder in the two-dimensional case renormalizes via the decimation of bonds and sites to a kind of randomness that is reminiscent of bond or site dilution, for which a percolation transition occurs where z can be shown to diverge [16]. Thus, bearing such a RG picture in mind, also in the fully connected random bond ferromagnet the critical point is governed by a percolation fix-point as the diluted case and the physics emerging here (quantum activated dynamics at the critical point, $z(\delta \to \infty)$ etc.) should be identical. Work in this direction is actually in progress [22].

4 The Bose glass: The Griffiths-McCoy phase of localized bosons

Concluding we should mention that similar phenomena are observed in another system: the disordered boson-Hubbard model for the superconductor-to-insulator transition in amorphous superconducting films [5] defined by the Hamiltonian

$$H = -t \sum_{\langle ij \rangle} (a_i^+ a_j + a_i a_j^+) + \frac{U}{2} \sum_i n_i^2 - \sum_i \mu_i n_i \tag{13}$$

where $\langle ij \rangle$ are nearest neighbor pairs on a square lattice, a_i^+ (a_i) are boson creation (annihilation) operators, $n_i = a_i^+ a_i$ counts the number of bosons at site i, U is the strength of an on-site repulsion and μ_i is a random chemical potential. The model (13) is usually considered in the "phase-only" approximation [23], where amplitude fluctuations in (13) are integrated out. Assuming that the complex field Φ has the simple form $\Phi_i = |\Phi_0| e^{i\hat{\phi}_i}$, where $\hat{\phi}_i$ is the phase operator conjugate to the number operator $\hat{n}_i$ with commutation relation $[\phi_i, n_j] = i\delta_{ij}$, one gets the quantum-phase Hamiltonian

$$H_{QPH} = -K \sum_{\langle i,j \rangle} \cos(\hat{\phi}_i - \hat{\phi}_j) + \frac{U}{2} \sum_i \hat{n}_i^2 - \sum_i (\mu + v_i) \hat{n}_i \,, \tag{14}$$

where $\hat{n}_i$ now measures the deviation from a mean density. Fig. 4 shows a schematic phase diagram of this quantum phase model (14) in the case of weak disorder $v_i \in [-\Delta, +\Delta]$ with $\Delta < 0.5$.

A phase very similar to the Griffiths-McCoy phase in magnetic systems occurs here, it is the Bose glass phase that is not superfluid/superconducting any more due to disorder induced localization of the bosons but still compressible (in contrast to the Mott-insulator). The one-particle Greens-function $G_{\rm loc}(\tau) = [\langle a_i(\tau) a_i^+(0) \rangle]_{\rm av} \sim [\langle e^{i[\hat{\phi}_{\mathbf{r}}(\tau') - \hat{\phi}_{\mathbf{r}}(\tau)]} \rangle]_{\rm av}$ is actually also singular in this phase. The probability distribution of the local (zero-frequency) superfluid susceptibility $\chi_{loc} = \int_0^\beta d\tau G_i(\tau)$ has an algebraic tail, as can be demonstrated via quantum Monte-Carlo simulations, see Fig. 4. This is completely equivalent to the power-law distributions encountered in the Griffiths-McCoy phase in the magnetic systems, however, the exponent does not change here: $z(K) = 2 = z_{\rm crit}$ throughout the Bose glass phase.

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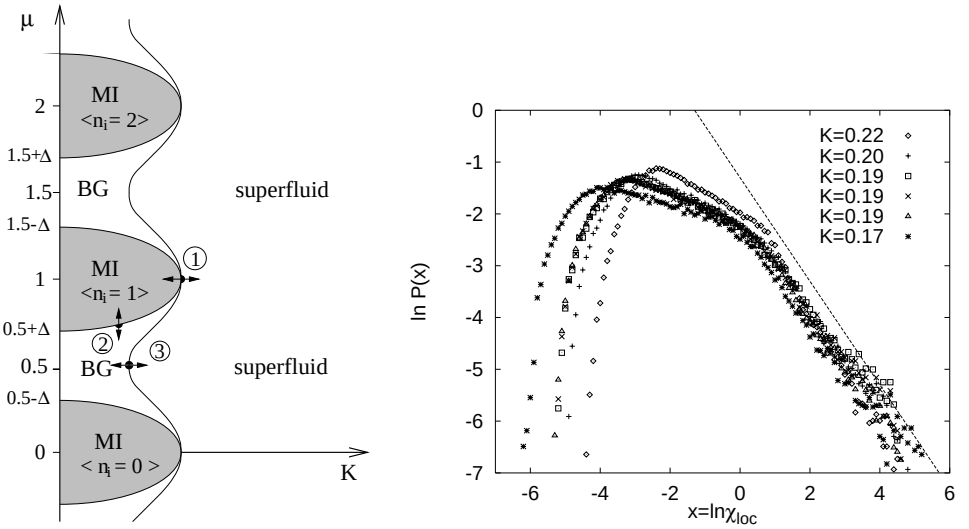

Fig. 4 Left: The $T = 0$ phase diagram of the quantum phase model (14) with $U = 1$. The Mott lobes are centered around integer values of the chemical potential. The Hamiltonian (14) is periodic in the chemical potential with period one. The "ordered" phase is the superfluid or superconducting phase, the Mott-lobes are insulating and incompressible, the Bose glass phase is insulating but compressible. Point 1 at the tip of the lobes denotes a multicritical point, the point 3 is the generic Bose glass to superfluid transition, which is actually a localization transition. At point 2 the gap closes when entering the Bose glass phase due to local low-energy excitations. Right: The probability distribution $P(\ln \chi_{\rm loc})$ of the local susceptibility for various values of $K < K_C = 0.247$ in the Bose glass phase ($\Delta = 0.5, \mu = 0.5$). The system size is $L = 6$ and $\beta = 200$. For $K = 0.19$, also data for $L = 4$ and $L = 10$ are shown, which is indistinguishable from $L = 6$. The dashed line has slope -1, which implies $d/z(K) = 1$, i.e. $z(K) = d(= z_{\rm crit}) = 2$.

References

[1] R.B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969)
[2] B.M. McCoy, Phys. Rev. Lett. **23**, 383 (1969)
[3] M. J. Thill and D. A. Huse, Physica A **15**, 321 (1995); H. Rieger and A. P. Young, Phys. Rev. B **54**, 3328 (1996); M. Guo, R. N. Bhatt and D. A. Huse, Phys. Rev. B **54**, 3336 (1996)
[4] W. Wu, B. Ellmann, T. F. Rosenbaum, G. Aeppli and D. H. Reich, Phys. Rev. Lett. **67**, 2076 (1991); W. Wu, D. Bitko, T. F. Rosenbaum and G. Aeppli; Phys. Rev. Lett. **71**, 1919 (1993)
[5] M. P. A. Fisher, P. B. Weichman, G. Grinstein and D. S. Fisher, Phys. Rev. B. **40**, 546 (1988). Y. Liu et al., Phys. Rev. Lett. **67**, 2068 (1991); M. A. Palaanen, A. F. Hebard and R. R. Ruel, Phys. Rev. Lett. **69**, 1604 (1992)
[6] P. A. Crowell, F. W. Van Keuls and J. D. Reppy, Phys. Rev. Lett. **75**, 1106 (1995); P. A. Crowell et al., Phys. Rev. B **51**, 12721 (1995)
[7] R. T. Scalettar, G. G. Batrouni and G. T. Zimanyi, Phys. Rev. Lett. **66**, 3144 (1991); W. Krauth, N. Trivedi and D. Ceperley, Phys. Rev. Lett. **67**, 2307 (1991); M. Wallin, E. S. Sørensen, S. M. Girvin and A. P. Young, Phys. Rev. B **49**, 12115 (1994); J. Kisker and H. Rieger, Phys. Rev. B **55**, 11981R (1997); Physica A **246**, 348 (1997)
[8] A. H. Castro Neto, G. Castilla and B. A. Jones, Report cond-mat/9710123; M. C. Andrade et al., Report cond-mat/9802081
[9] D.S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); Phys. Rev. B **51**, 6411 (1995)

[10] A. P. Young and H. Rieger, Phys. Rev. B **53**, 8486 (1996); F. Iglói and H. Rieger, Phys. Rev. Lett. **78**, 2473 (1997); F. Iglói and H. Rieger, Phys. Rev. B **57**, 11404 (1998)
[11] H. Rieger and F. Iglói, Europhys. Lett. **39**, 135 (1997)
[12] M. Randeria, J. P. Sethna and R. G. Palmer, Phys. Rev. Lett. **54** (1985) 1321; A. J. Bray, Phys. Rev. Lett. **54**, (1988) 720
[13] H. Rieger and A. P. Young, Phys. Rev. Lett. **72**, 4141 (1994);
M. Guo, R. N. Bhatt and D. A. Huse, Phys. Rev. Lett. **72**, 4137 (1994)
[14] F. Iglói and H. Rieger, Phys. Rev. E **58**, 4238 (1998)
[15] H. Rieger and N. Kawashima, Phys. Rev. Lett., submitted (1998), Report cond-mat/9802104
[16] T. Senthil and S. Sachdev, Phys. Rev. Lett.**77**, 5292 (1996);
T. Ikegami, S. Miyashita and H. Rieger, J. Phys. Soc. Jap. **67**, 2761 (1998)
[17] B. B. Beard, U. J. Wiese, Phys. Rev. Lett. **77**, 5130 (1996)
[18] N. V. Prokof'ev, B. V. Svistunov, I. S. Tupistyn, JETP Letters **64**, 911 (1996);
Report cond-mat/9703200 (1997)
[19] M. Suzuki, Progr. Theor. Phys. **56**, 1454 (1976)
[20] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. **58**, 86 (1987)
[21] H. Rieger and N. Kawashima, in preparation,
C. Pich and A. P. Young, Report cond-mat/9802108
[22] N. Kawashima and H. Rieger, in preparation;
D. Huse and D. Fisher, private communication
[23] M. P. A. Fisher, G. Grinstein, Phys. Rev. Lett. **60**, 208 (1988)