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Critical exponents of random XX and XY chains: Exact results via random walks

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Abstract. We study random XY and (dimerized) XX spin-1/2 quantum spin chains at their quantum phase transition driven by the anisotropy and dimerization, respectively. Using exact expressions for magnetization, correlation functions and energy gap, obtained by the free fermion technique, the critical and off-critical (Griffiths-McCoy) singularities are related to persistence properties of random walks. In this way we determine exactly the decay exponents for surface and bulk transverse and longitudinal correlations, correlation length exponent and dynamical exponent.

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Disordered quantum spin chains have gained much interest recently [1–8]. It seems to be established right now that the critical properties in these one-dimensional system are governed by an infinite-disorder fixed-point [9] and the application of a renormalization group (RG) scheme à la Dasgupta and Ma [10] is a powerful tool to determine critical properties and static correlations of these new universality classes, either analytically — if possible — or numerically. Although the underlying renormalization scheme is extremely simple the analytical computations are sometimes tedious [1,2]. Therefore an alternative route to the exact determination of critical exponents and other quantities of interest is highly desirable, and this is what we are going to present in this letter. In doing so we follow a route on which we already traveled successfully for the random transverse Ising chain [11–13], and here we are going to do one step further studying random XX and XY models with the help of a straightforward and efficient mapping to random walk problems. This mapping is not only a short-cut to the results known from analytical RG calculations, it also gives new exact results in the off-critical region (the Griffiths-phase [14]) and provides a mean to study situations in which the RG procedure must fail, as for instance in the case of correlated disorder [15], where the distribution is not invariant under renormalization. Here we confine ourselves to a concise presentation of the basic ideas including the determination of various exponents for the first time. The technical details of the derivations and further results are deferred to a subsequent publication [16].

The model that we consider is a spin-1/2 XY–quantum spin chain with L sites and open boundaries, defined by the Hamiltonian

$$H = \sum_{l=1}^{L-1} \left(J_l^x S_l^x S_{l+1}^x + J_l^y S_l^y S_{l+1}^y \right), \tag{1}$$

where the $S_l^{x,y}$ are spin-1/2 operators and the interaction strengths or couplings $J_l^{x,y} > 0$ are independent random variables modeling quenched disorder. In the case of the random XY chain one has two independent distributions for the couplings J^x and J^y, ρ^x and ρ^y, respectively, whereas the random dimerized XX-chain has perfectly isotropic couplings $J_l^x = J_l^y = J_l$ but two independent probability distributions for the even and odd couplings (*i.e.* for $J_{2l} = J_{2l}^e$ and $J_{2l-1} = J_{2l-1}^o$), ρ^e and ρ^o, respectively.

The model (1) has a critical point given by $[\ln J^x]_{\rm av} = [\ln J^y]_{\rm av}$ in the XY case [2] and $[\ln J^e]_{\rm av} = [\ln J^o]_{\rm av}$ in the XX case [2,17] (here $[\ldots]_{\rm av}$ denotes the disorder average). The distance from the critical point is conveniently measured in the variable

$$\delta = \frac{[\ln J^{x(e)}]_{\rm av} - [\ln J^{y(o)}]_{\rm av}}{\mathrm{var}[\ln J^{x(e)}] + \mathrm{var}[\ln J^{y(o)}]}, \tag{2}$$

where $\mathrm{var}(x)$ is the variance of random variable x. At the critical point ($\delta = 0$) spatial correlations decay

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Table 1. Finite-size exponents of the transverse and longitudinal order-parameters of the random XY and XX models.

	x^x–transverse		x^z–longitudinal	
	XY	XX	XY	XX
bulk	$(3-\sqrt{5})/2$	1	2	1
surface	1/2	1/2	1	1/2

algebraically, for instance in a finite system of length L with *periodic* boundary conditions the bulk-correlations decay as

$$[C^\mu(L)]_{\rm av} = [\langle 0|S_1^\mu S_{L/2}^\mu|0\rangle]_{\rm av} \sim L^{-\eta^\mu} \qquad (3)$$

for $\mu = x, y, z$, $\langle 0|$ denotes the ground state of (1), whereas for a finite system of length L with *open* boundary conditions the end-to-end correlations decay with a different exponent like

$$[C_1^\mu(L)]_{\rm av} = [\langle 0|S_1^\mu S_L^\mu|0\rangle]_{\rm av} \sim L^{-\eta_1^\mu}. \qquad (4)$$

Away from the critical point ($\delta \neq 0$) the infinite system develops long range order. For the XY model $\lim_{L\to\infty}[C^\mu(L)]_{\rm av} = (m^\mu)^2 \neq 0$, with $m^x > 0$ for $\delta > 0$ and $m^y > 0$ for $\delta < 0$, whereas for the XX model there is dimerization for $\delta \neq 0$ with non-vanishing string order [17]. One can introduce local transverse and longitudinal order parameters $m_l^{x,y}$ and m_l^z also for a finite system (with open boundaries) using the off-diagonal matrix element $[m_l^\mu]_{\rm av} = [\langle 1|S_1^\mu|0\rangle]_{\rm av}$, where $\langle 1|$ is the lowest excited state with a non-vanishing matrix-element [18]. Analogous to bulk and end-to-end correlations the bulk and surface magnetizations $m_{L/2}^\mu$ and m_1^μ, respectively, behave differently. At the critical point:

$$[m_{L/2}^\mu]_{\rm av} \sim L^{-x^\mu} \quad \text{and} \quad [m_1^\mu]_{\rm av} \sim L^{-x_1^\mu} \quad (\delta = 0) \qquad (5)$$

where the critical exponents x^μ and x_1^μ fulfill the scaling relation

$$2x^\mu = \eta^\mu \quad \text{and} \quad 2x_1^\mu = \eta_1^\mu. \qquad (6)$$

We have calculated the finite size exponents, which are related through scaling relations in (6) to the decay exponents and have collected in Table 1. In the following we show how these results can be obtained.

First, we transform the Hamiltonian in (1) by standard techniques [19] into a free-fermion problem:

$$H = \sum_{q=1}^{L} \epsilon_q \left(\eta_q^+ \eta_q - \frac{1}{2}\right), \qquad (7)$$

where the energy of modes, $\epsilon_q \geq 0$, for open boundary conditions (b.c.) can be most conveniently obtained [16] through the diagonalization of a symmetric, tridiagonal matrix, $\mathbf{T}$, of size $2L \times 2L$. The non-vanishing elements of $\mathbf{T}$ in its upper half are $T_{4i-3,4i-1} = J_{2i-1}^y$, $T_{4i-2,4i} = J_{2i-1}^x$, $T_{4i-1,4i+1} = J_{2i}^x$, and $T_{4i,4i+2} = J_{2i}^y$, whereas the corresponding eigenvectors, $\mathbf{V}_q$, are normalized as $|\mathbf{V}_q|^2 = 2$.

The components of the first eigenvector, $V_1(l)$, enter into the expression of two order-parameters: to the longitudinal magnetization of the XX model $m_{2i-1}^z(XX) = [V_1(2i-1)]^2/2$ and to the surface transverse magnetization of the XX and XY models $m_1^x = V_1(1)/2$. If the energy of the first excitation is vanishing, $\epsilon_1 = 0$, (which is the case in the thermodynamic limit in the ordered phase or in a finite system with fixed spin b.c., $S_L^x = \pm 1/2$, which amounts to have $J_{L-1}^y = 0$,) the above order parameters can be explicitly expressed through the couplings as [16]:

$$m_{2l-1}^z(XX) = \frac{1}{2}\left\{1 + \sum_{k=l}^{L/2-1}\prod_{j=l}^{k}\left(\frac{J_{2j-1}}{J_{2j}}\right)^2 + \sum_{k=1}^{l-1}\prod_{j=1}^{k}\left(\frac{J_{2l-2j}}{J_{2l-2j-1}}\right)^2\right\}^{-1} \qquad (8)$$

and

$$m_1^x = \frac{1}{2}\left[1 + \sum_{l=1}^{L/2-1}\prod_{j=1}^{l}\left(\frac{J_{2j-1}^{y(o)}}{J_{2j}^{x(e)}}\right)^2\right]^{-1/2}, \qquad (9)$$

where for convenience we assume from now on that L is a multiple of 4.

For the surface ($l = 1$) $m_1^z(XX)$ in (8) and m_1^x in (9) are closely related and they are similar to an analogous result for the random transverse Ising chain [11]. The finite-size scaling properties of their average are related to the survival probability of a random walk with $L/2$ steps. For $m_1^z(XX)$ this is easy to see for the extreme binary distribution, in which $J_{2j} = 1$ and $J_{2j-1} = \lambda$, λ^{-1} with probability 1/2, taking the limit $\lambda \to 0$ (*i.e.* $\lambda^{-1} \to \infty$). Due to the occurrence of infinite terms in the sum in the denominator of the r.h.s. of (8) one can easily identify those instances that give a non-vanishing surface magnetization: When $\forall k = 1, \ldots, L/2-1 : \prod_{j=1}^{k} J_{2j-1} < \infty$ the expression on the r.h.s. of (8) attains a non-vanishing value (typically 1 or, less frequently, some fraction $1/n$), otherwise it is zero. One can represent the disorder configuration $J_1, J_3, J_5, \ldots, J_{L-1}$ as one instance of a random walk with $L/2-1$ steps by saying that the walker in the i-th steps moves downwards if $J_{2i-1} = \lambda$ and upwards if $J_{2i-1} = \lambda^{-1}$, as it is sketched in Figure 1. In this way the disorder configuration with non-vanishing surface magnetization m_1^z are easily identified: they represent surviving walks, *i.e.* walks that never move into the upper half. Thus $[m_1^z(XX)]_{\rm av}$ scales like the survival probability $P_{\rm surv}(L/2)$ of a random walk with $L/2$ steps that vanishes like $L^{-1/2}$, *i.e.* $[m_1^z(XX)]_{\rm av} \sim L^{-1/2}$ and $x_1^z(XX) = 1/2$, as given in Table 1. Analogous analysis of the expression in (9) for the surface transverse order-parameter leads to identical result: $x_1^x(XX, XY) = 1/2$.

For the bulk longitudinal order-parameter inspecting the expression (8) for $l = L/4$ one sees that now (again for the extreme binary distribution) a non-vanishing magnetization $m_{L/2-1}^z$ arises only if $\forall k = L/4, \ldots, L/2-1 : \prod_{j=L/4}^{k} J_{2j-1} < \infty$ and $\forall k = 1, \ldots, L/4-1 : \prod_{j=1}^{k} J_{L/2-2j-1}^{-1} < \infty$. We represent the disorder configuration to the right of the central site, $J_{L/2-1}, J_{L/2+1}, \ldots, J_{L-1}$, as a random walk with $L/4$ steps in the way as for the surface spin. The disorder

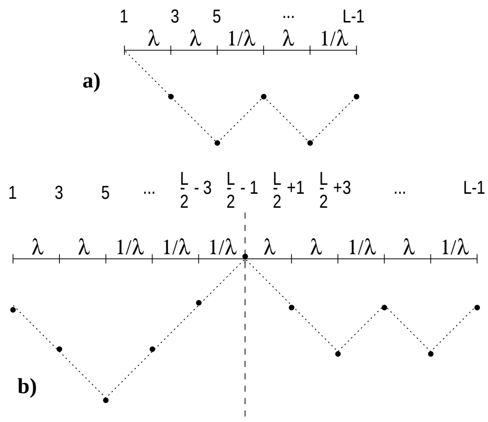

Fig. 1. Sketch of the configuration of odd bonds for a chain of length L that gives a non-vanishing longitudinal magnetization $m_i^z \sim \mathcal{O}(1)$ for the surface spin, $i=1$, in (a) and the central spin, $i = L/2-1$, in (b). The example is for the extreme binary distribution with $J_{2i}=1$. Weak couplings ($J_{2i-1}=\lambda$) correspond to downward steps of the random walk, strong couplings ($J_{2i-1}=\lambda^{-1}$) to upwards steps. The walk in (a) has surviving character, it does not enter the upper half plane. In (b) one can identify two random walks each starting at the central site, $i=L/2-1$, one to the right and one to the left, and each of them has surviving character.

configuration $J_{L/2-1}, J_{L/2-3}, \dots, J_1$ to the left is represented as a second (independent) random walk also with $L/4$ steps, now counting backwards and with the step-direction reversed (*i.e.* downwards for $J=\lambda^{-1}$ and upwards for $J=\lambda$), since now strong bonds on odd sites imply weak coupling of the central spin. For illustration this representation is depicted in Figure 1. Now, for the bulk magnetization $m^z_{L/2-1}$ to be non-vanishing, *both* halfs of the coupling configuration have to represent surviving random walks. Thus the probability for a non-vanishing magnetization $m^z_{L/2-1}$ is just the product of two survival probabilities (since both walks are independent), *i.e.* $[m^z_{L/2-1}]_{\rm av} \sim \{P_{\rm surv}(L/4)\}^2 \sim L^{-1}$ and therefore $x^z(XX)=1$.

The scaling behaviour of the longitudinal magnetization for the XY chain follows from the exact relation [16]

$$[m_l^z(XY)]_{\rm av} = [\{m_l^z(XX)\}^{1/2}]^2_{\rm av}. \qquad (10)$$

Since $\{m_1^z(XX)\}^{1/2}$ has a non-vanishing value if and only if $m_1^z(XX)$ is non-vanishing, one obtains $[m_1^z(XY)]_{\rm av} \sim \{P_{\rm surv}(L/4)\}^2 \sim L^{-1}$ and $[m^z_{L/2-1}(XY)]_{\rm av} \sim \{P_{\rm surv}(L/4)\}^4 \sim L^{-2}$, from which the exponents in Table 1 can be obtained.

For the transverse bulk order parameter in the XY chain we use the fact that the model can be mapped onto two transverse Ising models (TIM), with uncorrelated disorder in both chains [2,16]. Through this mapping one obtains for the transverse correlation function $C^x_{2i,2i+2r} = \langle 0|S^x_{2i}S^x_{2i+2r}|0\rangle$ the following identity [16]

$$[C^x_{2i,2i+2r}(XY)]_{\rm av} = 4[C^x_{i,i+r}(\mathrm{TIM_{free}})]_{\rm av} \times [C^x_{i,i+r}(\mathrm{TIM_{fixed}})]_{\rm av} \sim r^{-2\eta^x(\mathrm{TIM})}, \qquad (11)$$

Table 2. Surface and bulk transverse order-parameters averaged over 50000 surviving walk coupling configurations for the binary distribution ($\lambda = 0.1$).

L	$2[m_1^x]_{\rm sw}$	$2[m^x_{L/2}]_{\rm sw}$
32	0.994	0.764
64	0.991	0.682
128	0.991	0.647
256	0.991	0.577

where *fixed* and *free* indicated the boundary conditions. Since the correlation function exponent of the TIM is known exactly [1] we have $x^x(XY) = \eta^x(\mathrm{TIM}) = (3-\sqrt{5})/2$.

For the XX chain the two transverse Ising chains have perfectly correlated disorder, which implies that the disorder averaged transverse correlations do not factorize into two independent averages as in (11). Therefore, for the transverse order parameter exponent in the XX case we have to use a different route: The first important observation is that the transverse bulk order parameter $m^x_{L/2} = \langle 1|S^x_{L/2}|0\rangle$ attains its maximum value 1/2 if the central spin is decoupled from the rest of the system, *i.e.* when $J_{L/2-1} = J_{L/2} = 0$. More generally we expect that $m^x_{L/2} \sim \mathcal{O}(1)$ when it is weakly coupled to the rest of the system. "Weakly coupled" in the case of the extreme binary distribution means that the bond configuration to the left and to the right of the central spin represent both surviving random walks, as exemplified in Figure 1b (this is actually equivalent to the (exact) condition for the longitudinal order parameter $m^z_{L/2}(XX)$ to be non-vanishing). This correspondence implies $[m^x_{L/2}(XX)]_{\rm av} \sim \{P_{\rm surv}(L/4)\}^2 \sim L^{-1}$ from which one obtains $x^x(XX)=1$.

We verified the strong correlation between weak coupling and non-vanishing transverse order parameter numerically in the following way: We considered a chain with $L+1$ sites and the couplings at both sides of the central spin were taken randomly in the form of surviving walk character, where we used the binary distribution with $\lambda = 0.1$. For such small value of λ the surface order-parameter averaged over the surviving walk (sw) configurations $[m_1^x]_{\rm sw}$ was very close to the maximal value of 1/2. Then we calculated numerically the order-parameter at the central spin and its average value over surviving walk configurations $[m^x_{L/2}]_{\rm sw}$ as given in Table 2.

As seen in the Table the averaged surface order-parameter stays constant for large values of L, whereas the bulk order-parameter decreases very slowly, actually slower than any power. The data can be fitted by $[m^x_{L/2}]_{\rm sw} \sim (\ln L)^{-\sigma}$, with $\sigma \approx 0.5$. Thus we conclude that $[m^x_{L/2}]_{\rm av} \sim \{P_{surv}(L/4)\}^2 [m^x_{L/2}]_{\rm sw}$ and the numerical results confirm the exponent given in Table 1, however there are strong logarithmic corrections, which imply for the average transverse correlations

$$[C^x(r)]_{\rm av} \sim r^{-2}\ln^{-1}(r) \quad XX\text{-model}. \qquad (12)$$

These strong logarithmic corrections render the numerical calculation of critical exponents very difficult [17,20]. In earlier numerical work using smaller finite systems disorder dependent exponents were reported [20]. We believe that these numerical results can be interpreted as effective, size-dependent exponents and the asymptotic critical behavior is indeed described by equation (12).

Away from the critical point the correlation length exponent ν can be determined by the scaling behavior of the longitudinal surface magnetization, *i.e.* (8) with $l = 1$, which can be inferred from the survival properties of a, now biased, random walk: $[m_1^{x,y}(\delta, L)]_{\rm av} \sim P_{\rm surv}(\delta, L/2)$ [6].

The characteristic length scale ξ of surviving walks corresponds to the *average* correlation length of the XX and XY chains and is given by $[\xi]_{\rm av} \sim \delta^{-\nu}$ with $\nu = 2$.

The typical correlation length, $\xi_{\rm typ}$ can be inferred from the scaling behavior of the typical surface magnetization $\ln m_1 \sim \sum_j \{\ln(J_{2j-1}^{y(o)}) - \ln(J_{2j}^{x(e)})\} \propto \delta L$, which gives $[\xi]_{\rm typ} \sim \delta^{-\nu_{\rm typ}}$ with $\nu_{\rm typ} = 1$.

The critical and off-critical scaling behavior of the low energy excitations and dynamical correlations can be deduced from a formula for the gap ϵ_1 [16]

$$\epsilon_1(L) \sim m_1^x m_{L-1}^x J_{L-1}^y \prod_{j=1}^{L/2-1} \frac{J_{2j-1}^{y(o)}}{J_{2j}^{x(e)}}, \tag{13}$$

which can be obtained in a perturbative solution of the equation $\mathbf{T}V_1 = \epsilon_1 V_1$ and m_{L-1}^x denotes the order-parameter at the other end of the chain. At the critical point one observes that $\ln \epsilon_1$ is a sum of L independently distributed random variables with zero mean (since $\delta = 0$), for which the central limit theorem applies. Therefore the probability distribution of gaps obeys $P(\ln \epsilon_1) \sim L^{-1/2} \tilde{p}(\ln \epsilon_1 / L^{-1/2})$ and one uses scaling arguments as in [21] to deduce the asymptotic (imaginary) time dependence of the spin-spin autocorrelation function $G_l^\mu(\tau) = [\langle 0 | S_l^\mu(\tau) S_l^\mu(0) | 0 \rangle]_{\rm av}$.

$$G_a^\mu(\tau) \sim (\ln \tau)^{-\eta_a^\mu} \tag{14}$$

for the surface ($a = 1$) and bulk ($a =$ bulk), respectively, with the critical exponents η_a^μ as given above.

Away from the critical point in the Griffiths-phase [14] the gap distribution has still an algebraic tail $P(\epsilon_1) \sim \epsilon_1^{-1+1/z'(\delta)}$, with a dynamical exponent $z'(\delta)$ that varies continuously with the distance from the critical point δ and is given by the exact (implicit) formula [12]

$$\left[\left(\frac{J^{x(e)}}{J^{y(o)}} \right)^{1/z'(\delta)} \right]_{\rm av} = 1. \tag{15}$$

The dynamical exponent $z'(\delta)$ parameterizes all Griffiths-McCoy singularities occurring in the Griffiths-phase, *e.g.* the spin-spin autocorrelations decay algebraically as

$$G_l(\tau) \sim \tau^{-1/z'(\delta)}, \tag{16}$$

which gives for the susceptibility $\chi^\mu \sim T^{-1+1/z'(\delta)}$ diverging for $T \to 0$ ($T =$ temperature) if $z'(\delta) > 1$.

To summarize we have shown how to obtain the critical and off-critical singularities of random XX and XY chains with simple random walk arguments using exact formulas arising from the free fermion description of these quantum spin models. All results for the critical exponents are therefore exact. One should note that for the transverse bulk order parameter exponent for the XY model we referred to a result for the transverse Ising model obtained by a RG calculation [1] and for the same exponent of the XX model we showed the existence of strong logarithmic corrections.

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References

1. D.S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); Phys. Rev. B **51**, 6411 (1995).
2. D.S. Fisher, Phys. Rev. B **50**, 3799 (1994).
3. A.P. Young, H. Rieger, Phys. Rev. B **53**, 8486 (1996); F. Iglói, H. Rieger, Phys. Rev. Lett. **78**, 2473 (1997).
4. R.A. Hyman, K. Yang, R.N. Bhatt, S.M. Girvin, Phys. Rev. Lett. **76**, 839 (1996); Kun Yang, R.N. Bhatt, Phys. Rev. Lett. **80**, 4562 (1998).
5. R.H. McKenzie, Phys. Rev. Lett. **77**, 4804 (1996); J.E. Bunder, R.H. McKenzie, Phys. Rev. B **60**, 344 (1999).
6. M. Fabrizio, R. Mélin, Phys. Rev. Lett. **78**, 3382 (1997).
7. C. Monthus, O. Golinelli, Th. Jolicoeur, Phys. Rev. Lett. **79**, 3254 (1997).
8. E. Westerberg, A. Furusaki, M. Sigrist, P.A. Lee, Phys. Rev. B **55**, 12578 (1997); T. Hikihara, A. Furusaki, M. Sigrist, Phys. Rev. B **60**, 12116 (1999).
9. D.S. Fisher, Physica A **263**, 222 (1999).
10. C. Dasgupta, S.K. Ma, Phys. Rev. B **22**, 1305 (1979).
11. F. Iglói, H. Rieger, Phys. Rev. B **57**, 11404 (1998).
12. F. Iglói, H. Rieger, Phys. Rev. E **58**, 4238 (1998).
13. H. Rieger, F. Iglói, Europhys. Lett. **45**, 673 (1999).
14. R.B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969); B.M. McCoy, Phys. Rev. Lett. **23**, 383 (1969); Phys. Rev. **188**, 1014 (1969).
15. H. Rieger, F. Iglói, Phys. Rev. Lett. **83**, 3741 (1999).
16. F. Iglói, R. Juhász, H. Rieger, in preparation (1999).
17. P. Henelius, S.M. Girvin, Phys. Rev. B **57**, 11457 (1998).
18. L. Turban, F. Iglói, J. Phys. A **30**, L105 (1997).
19. E. Lieb, T. Schultz, D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).
20. H. Röder, J. Stolze, R.N. Silver, G. Müller, J. Appl. Phys. **79**, 4632 (1996).
21. H. Rieger, F. Iglói, Europhys. Lett. **39**, 135 (1997).