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Ground states versus low-temperature equilibria in random field Ising chains

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Abstract. Random walk arguments and exact numerical computations are used to study one-dimensional random field chains. The ground state structure is described with absorbing and non-absorbing random walk excursions. At low temperatures, the local magnetization follows the ground state except at regions where a local random field fluctuation makes thermal excitations easier. This is explained by the random walk picture, implying that the magnetization lengthscale is a product of the domain size and the thermal excitation scale.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 75.50.Lk Spin glasses and other random magnets

In statistical mechanics of random systems the search for universal properties has a geometric interpretation. If the introduction of disorder is relevant, the real-space properties of the physical states can be understood through scaling exponents. These describe the fluctuations of a domain wall, or the behavior of a spin-spin correlation function, and the configurational energy is coupled to the geometry. A domain wall in a magnet wanders in space, described by a roughness exponent ζ and there is an exponent θ for the free or ground state energy fluctuations (see [1]). Assuming that the 'zero temperature fixed point' scenario is true and that the entropy is irrelevant at low enough temperatures, this is all what is needed. The system evolves *via* Arrhenius-like dynamics so that the cost of moving in the energy landscape is given by the exponential factor $\exp(\Delta E\beta)$, where $\beta = 1/T$ and T is the temperature, and $\Delta E \sim l^{\zeta}$ relates the cost to the scale of the perturbation l. A simple toy model, the random energy model, attempts to describe the landscape but does not have the spatial structure that is crucial in finite-dimensional systems [2]. Similar physics arises in systems with structural frustration, only, like glasses.

A random magnet has a ground state (GS), described exactly by the positions and arrangement of the domain walls. Non-trivial examples abound in the form of Ising spin glasses and random field Ising magnets [3]. Here we develop a novel random walk (RW) picture, that allows us to solve exactly for the groundstate of one-dimensional random field chains, generalizing earlier applications of RW ideas to this system [4]. We compare, by considering the local magnetizations, this solution to exact numerical computations at $T > 0$ to discuss the relation of the low-temperature physics to the GS of a random magnet. Our work is analogous to RW arguments in disordered quantum spin chains, where one can also compute exactly physical properties [5].

The GS structure can be constructed for arbitrary field distributions *via* the random walk 'algorithm' (see below). At finite temperatures we present a scaling argument based on the zero-temperature description of the energy landscape, and confirm it by numerical studies of the GS and the local magnetization on a sample-to-sample basis. The second main finding is then the existence of two length scales. These are the zero-temperature length scale of the domains and the typical size of 'easy' excitations at a given temperature. The latter changes the correlation length of the magnetization from the GS. The emerging picture should be applicable to more general situations than the 1D RFIM. This is the simplest random magnet where a non-trivial GS is mixed with thermal excitations (*e.g.* random bond Ising magnets have a trivial GS).

The non-equilibrium properties of the RFIM chain have recently received attention [6,7] since decimation-type real space renormalization can be applied to domain wall dynamics: each DW undergoes logarithmic Sinai diffusion [8]. The asymptotic state of *e.g.* coarsening (on which the related RG procedure is to be stopped) is given by our findings. The zeroes of the magnetization profile simply denote the equilibrium positions of domain walls

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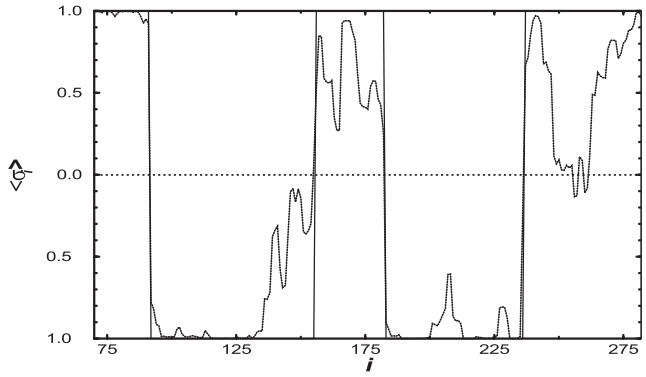

Fig. 1. Segment of a ground state configuration (bold line), compared to the equilibrium local magnetization $\langle\sigma_i\rangle$ at temperature $T = 0.45J$. In the leftmost domain little melting occurs, the one around $i = 125$ is an example of boundary melting and in the rightmost one even spin reversal occurs.

at $T > 0$, with extra domain walls added to the GS due to magnetization reversals.

The RFIM Hamiltonian is

$$H = -\sum_{i=1}^{N} J\sigma_i\sigma_{i+1} + \sum_i h_i\sigma_i \tag{1}$$

where the σ's are spins located at sites i of the chain, and h_i are random fields from a suitable probability distribution $P(h_i)$ ($\langle h_i\rangle = 0$, with variance h_r). For a binary distribution $h_i = \pm h_{i,r}$ the model is equivalent to a spin glass chain (with couplings $J_i = \pm J$) in a homogeneous external field h_r [9,10]. Figure 1 shows typical GS and finite temperature ($T = 0.45$) magnetization profiles. The GS domain size is often thought to be given by the Imry-Ma argument [11]: the domain field energy balances the cost from the domain walls on a scale $[l]_{\rm av} \sim 1/h_r^2$ in 1D ($[\ldots]_{\rm av}$ denotes the disorder average). This reasoning omits the global optimization that yields the GS. At finite but small temperatures the magnetization changes as the GS domain walls fluctuate, and thus the $m(x)$-profile changes continuously. More interestingly, there are regions inside domains where the magnetization can undergo a local reversal, arising from the local field configurations as demonstrated below.

The starting point for the random walk argument is the fact that any sequence $\mathcal{S}$ of lattice sites i with $|\sum_{i\in\mathcal{S}} h_i| \geq 2J$ leads to a GS spin structure with $\sigma_i = +1 \quad \forall i \in \mathcal{S}$ if $\sum_{i\in\mathcal{S}} h_i \geq 2J$ (and $\sigma_i = -1 \quad \forall i \in \mathcal{S}$ if $\sum_{\rangle\in S} h_i \leq -2J$) independent of the local fields h_j at sites $j \notin \mathcal{S}$. The system can thus be split up into such *absorbing excursions* and into the remaining lattice sites, which make up so-called *non-absorbing excursions*. Figure 2 illustrates this: an *absorbing excursion* is a sequence of spins starting at some lattice site i and ending at the lattice site $j \geq i$, with the field-sum $|\sum_{i\in\mathcal{S}} h_i|$ for the first time becoming

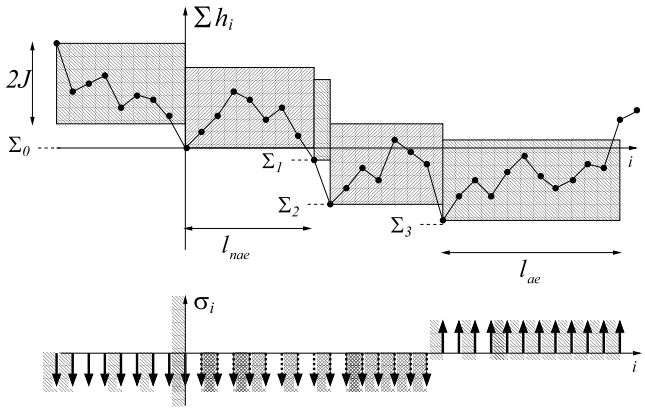

Fig. 2. The terminology used in the RW algorithm. For further details see text.

greater or equal to $2J$:

$$|\sum_{l=i}^{j} h_i| \geq 2J \qquad \text{and} \qquad |\sum_{l=i}^{k} h_i| < 2J \quad \forall i < k < j. \tag{2}$$

In Figure 2 the left- and rightmost sequences are absorbing excursions, of length $l_{\rm ae}$. A sequence $\mathcal{S}'$ of spins from i to $j \geq i$ is a *non-absorbing excursion* if

$$\overline{\sigma}\sum_{l=i}^{j} h_i \leq 0 \qquad \text{and} \qquad 0 < \overline{\sigma}\sum_{l=i}^{k} h_i < 2J \quad \forall i < k < j \tag{3}$$

where $\overline{\sigma} = \pm 1$ is the orientation of the spins within the preceding absorbing excursion. The length of a non-absorbing excursion is $l_{\rm nae}$. A simple 'step down' (like from $\sum_1$ to $\sum_2$) is included in this definition.

The GS now follows as a sequence of absorbing and non-absorbing excursions. It, and the Zeeman energy $E_f = \sum_{\rm domain} h_i$, and the mean domain-length can be determined with the rules: (1) determine an absorbing excursion $\mathcal{S}_0$ for a given field configuration. If it starts at site i_0, ends at j_0, and $\overline{\sigma}$ is the sign of its field-sum, then $\sigma_k = \overline{\sigma}$ for all $k \in \mathcal{S}_0$. (2) start from $j_0 + 1$ and find all $n_{\rm nae}$ non-absorbing excursions until the next absorbing excursion $\mathcal{S}_1$ (from i_1 to j_1) is found, whose field-sum is by definition opposite in sign to the preceding one. The sites k belonging to the non-absorbing excursions have the same orientation $\sigma_k = \overline{\sigma}$ as those within $\mathcal{S}_0$. The orientation of the spins at sites l within $\mathcal{S}_1$ is opposite to the latter one, $\sigma_l = -\overline{\sigma}$. (3) starting again at $j_1 + 1$ the search (2) for the next absorbing excursion then leads to the overall GS. These steps actually define a fast algorithm for finding the GS, though for historical reasons we have used the max-flow/min-cut solution [3].

The domain length consists of two contributions and the meanlength $[l_d]_{\rm av}$ is given by

$$[l_d]_{\rm av}(h_r) = [n_{\rm nae}]_{\rm av}[l_{\rm nae}]_{\rm av} + [l_{\rm ae}]_{\rm av}. \tag{4}$$

Now one needs to estimate the h_r-dependence of $[n_{\rm nae}]_{\rm av}$, $[l_{\rm nae}]_{\rm av}$ and $[l_{\rm ae}]_{\rm av}$. The fieldsum over the local fields of a non-absorbing excursion is a RW with absorbing boundaries at $\sum_{\rm nae} h_i = 0$ and $\sum_{\rm nae} h_i = 2J$ and random step size with zero mean and variance h_r. Rescaling the step size $h_i' = h_i/h_r \to 1$ this becomes a 1d RW starting from $x = 1$ at $t = 0$ with random step lengths with mean zero and variance one and absorbing boundaries at $x = 0$ and $x = L = 2J/h_r$. The probability $P_0(t, L)$ to be absorbed at $x = 0$ within the time interval $[t, t + {\rm d}t]$ without having been absorbed at $x = L$ reads $P_0(t, L) \propto t^{-3/2}$ for $1 \lesssim t \lesssim L^2$ and decays exponentially for $t \gtrsim L^2$. Integration over $P_0(t, L)$ leads to $[l_{\rm nae}]_{\rm av} \sim L \propto h_r^{-1}$. The mean number $[n_{\rm nae}]_{\rm av}$ of consecutive non-absorbing excursions follows from the probability for an excursion to be absorbing growing as $p_{\rm ae} \sim 1/L \propto h_r$ [12]. Thus $P(n_{\rm nae}) \sim (1 - p_{\rm ae})^{n_{\rm nae}}$ decays exponentially. As a consequence $n_{\rm nae} \sim 1/\ln(1 - b\,h_r)$ and the mean length of an absorbing excursion scales like h_r^{-2}. Finally equation (4) reads

$$[l_d]_{\rm av}(h_r) \sim \frac{a}{h_r\,\ln(1 - b\,h_r)} + \frac{c}{{h_r}^2} \quad \to \quad \frac{e}{{h_r}^2} \qquad \text{for} \quad h_r \to 0 \tag{5}$$

where one expects $b < 1$, $a < 0$ and $-a \approx c$. Note that $a/h_r\,\ln(1 - b\,h_r) \sim h_r^{-2}$ for $h_r \to 0$ and for $h_r < 0.5$ no significant difference between $a/[h_r \ln(1 - b\,h_r)]$ and h_r^{-2} can be observed. The asymptotic limit is Imry-Ma-like, though its origin is more complicated.

This result is confirmed by exact GS computations using a Gaussian random field distribution with zero mean and variance h_r and averaging over 10^5 disorder configurations. The system size is large enough ($L = 5000$) such that $[l_d]_{\rm av} \ll L$ even for the smallest field strength h_r. Figure 3 shows our numerical result for the average length $[l_d - 2]_{\rm av}$ of the GS domains as a function of the field amplitude h_r. In the limit $h_r \to \infty$ $[l_d]_{\rm av} \to 2$ since all the spins align with their local fields. In the limit $h_r \ll J$ the data fit well to the predicted form equation (5), scaling as h_r^{-2} for $h_r \to 0$. Moreover, as can be seen in Figure 4 the probability distribution of the domain sizes decays exponentially, with a decay rate ν that scales inversely proportional to $[l_d]_{\rm av}$, *i.e.* $\nu(h_r) \propto h_r^2$.

The field energy of a domain can be computed as a function of h_r and l_d by noting that both a single absorbing excursion and all of the non-absorbing excursions contribute. The former contributes a constant ($2J$), independent of h_r and l_d. Each non-absorbing excursion adds an amount of $\mathcal{O}(h_r)$ so that the sum self-averages. The contribution of a single non-absorbing excursion equals $\sum_i - \sum_{i-1} \sim h_r$, *i.e.* the step width of the RW. Thus the field energy results from the number of non-absorbing excursions in a domain, $n_{\rm nae}$, plus $2J$. From (5) we learn that in the limit $h_r \to \infty$ the contribution of the absorbing and non-absorbing walks to $[l_d]_{\rm av}$ scale similarly such that we expect that for a fixed domain size $[n_{\rm nae}(l_d)]_{\rm av} \propto l_d/[l_{\rm nae}]_{\rm av} \propto l_d h_r$. Thus one has

$$[E_f(l_d)]_{\rm av} = 2J + [n_{\rm nae}(l_d)]_{\rm av}\,h_r = 2J + d\,h_r^2\,[l_d]_{\rm av} \tag{6}$$

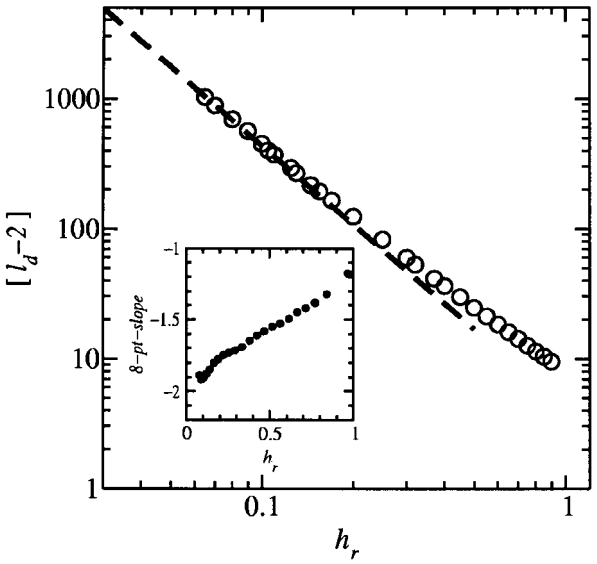

Fig. 3. Average domain length as a function of h_r. The dotted line is a fit to equation (5) with $a = -0.74$, $b = 0.25$ and $c = 1.4$. The inset shows the 8-point slope of the data yielding an exponent 2 in the limit of small h_r.

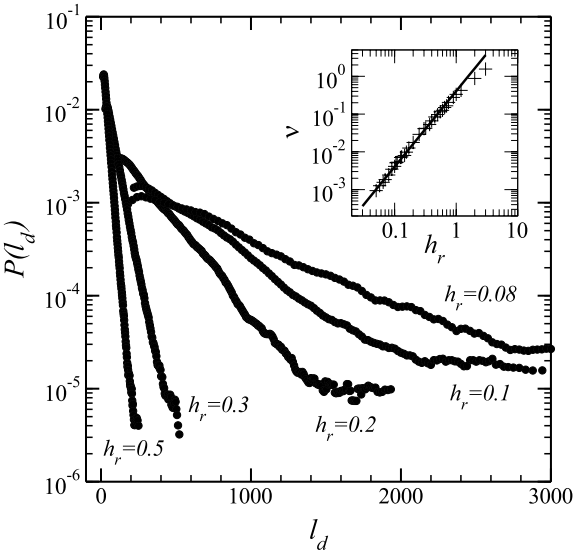

Fig. 4. Probability distribution of the domain lengths l_d. Apart from a non-exponential tail (perhaps due to finite size effects the decay is exponential with decay rate ν. Inset: The decay rate ν as a function of h_r. For $h_r \ll 1$ the data are compatible with $\nu \sim h_r^2$ (bold line).

The numerics confirms this result: Figure 5 shows the data for the mean Zeeman energy $[E_f(l_d)]_{\rm av}$ of domains of length l_d. From the slopes of the straight lines we see that $[E_f(l_d)]_{\rm av}$ is linear in the domain length and from the offsets that it grows like h_r^2, independent of the field distribution $P(h_i)$. Note that from a *naive* random walk picture one would expect $[E_f(l_d)]_{\rm av} \propto l_d^{1/2} h_r$.

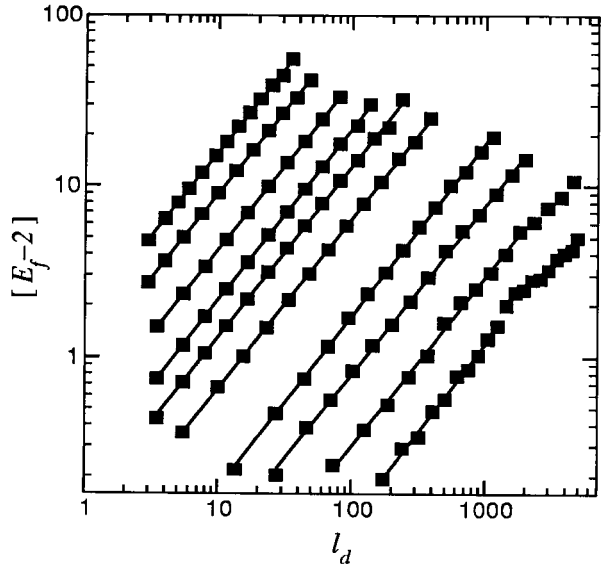

Fig. 5. Mean Zeeman energy $H_Z - 2J$ corresponding to a particular domain length l_d in a log-log plot. $h_r = 3.0, 2.0, 1.2, 1.0, 0.8, 0.6, 0.4, 0.2, 0.13, 0.08, 0.05$ from top to bottom. The slopes of the straight lines are all within the interval 1.00 ± 0.05. The data are averaged over 10^5 disorder configurations. The straight lines represent least square fits.

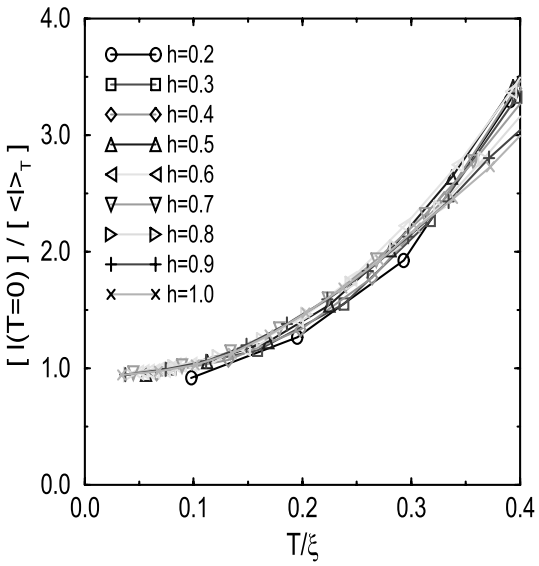

Fig. 6. Scaling-plot for the average domain-lengths at finite temperatures: $[l(T=0)]_{\rm av}/[\langle l_T \rangle]_{\rm T}$ *versus* $T/\xi(h)$ for different values of the field strength h_r, where the length scale $\xi(h) \propto h_r^{2/3}$.

We now turn our attention to $T > 0$ equilibria: the local magnetization $m(x)$ and the domain structure. Using numerical transfer matrix methods [13,14] to compute the partition function Z_N the exact expectation value $\langle \sigma_r \rangle$ for each spin is found by calculating the product of the N 2×2 transfer matrices. Floating point accuracy gives a lower limit of $T = 0.05$ due to the smallness of the random matrix elements. First we address the scale-lengths of the equilibrium magnetization by computing the average length $[l_m]_{\rm av}$ separating two consecutive zeroes of the magnetization. Figure 6 demonstrates how this length-scale changes with temperature, if we first scale away the $T = 0$-dependence on the field. A further collapse with the right combination of h_r and T leads to an universal scaling function for $[l_m]_{\rm av}$

$$[l_m]_{\rm av} = [l_d]_{\rm av} f(T/h_r^{2/3}), \tag{7}$$

where $f \to 1$ with $T \to 0$. The dependence of $[l_m]_{\rm av}$ on the combination of temperature and field strength does not follow an Imry-Ma-like scaling. It is determined by both zero-temperature effects ($[l_d]_{\rm av}$) and thermal fluctuations. The following argument explains the scaling variable $h_r^{2/3}/T$, analogously to spin glass chains in an external field [10]. The non-absorbing random walks (constituting the domains) are sometimes such that the random walk sum of fields over the excursion is close to $2J$. These almost-absorbing walks are the sequences (of spins) most likely to be flipped at finite temperatures. The cost of flipping such a part of a domain is proportional to J, which if measured in terms of h_r is related to the length-scale l of the non-absorbing excursions, $h_r \sim 1/l$. This is *almost* equal to the Zeeman-energy optimized over the excursion, which scales as $E_f \sim h_r l^{1/2}$. Equating the cost with the gain and solving for the energy scale (E_f) gives rise to the Arrhenius factor $E_f/T \sim h_r^{2/3}/T$.

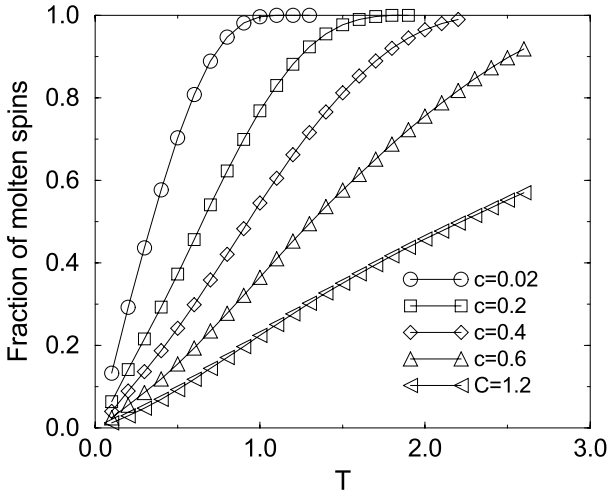

Fig. 7. Fraction of melted spins as a function of temperature for different values of c. $L = 400$, $h_r = 1.0$

As Figure 1 demonstrates, $m(x)$ for $T > 0$ differs from the GS due to domain wall fluctuations *and* internal cluster reversals. We introduce a parameter $c \in (0, 2)$ and define a reversal to be a sequence of spins for which $|\langle \sigma_i \rangle(T) - \sigma_i(T = 0)| > c$ holds; the definition is applied to both processes separately. Since bulk reversal involves the breaking of two extra bonds one expects that domain wall fluctuations dominate. However, the former contributes a considerable portion to the total melting even at low temperatures (Figs. 7 and 8). The relative portion of bulk

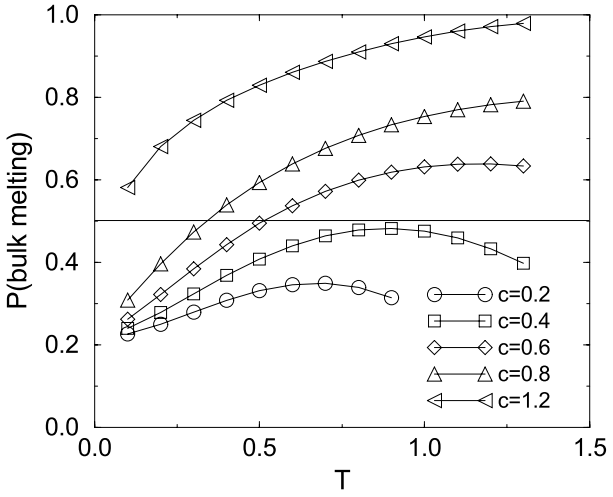

Fig. 8. Portion of those spins is displayed which reside inside bulk segments. Parameters as in Figure 7.

reversals at first grows with temperature for all values of c since the gain in entropy allows for more broken bonds. Moving the threshold c away from ± 1, a greater number of bulk segments is identified. Eventually for very large c even more bulk than boundary reversals are observed. The characteristic reversal rates are different for the two processes, and related *via* the empirical formula

$$\left(\frac{\Delta m}{\Delta T}\right)_{\text{bulk}} = \alpha \left(\frac{\Delta m}{\Delta T}\right)_{\text{bound}} , \qquad (8)$$

with $\alpha \approx 1.63$. Thus the change in magnetization with increasing T is stronger inside the GS domains than at their boundaries, independent of h_r [15].

In conclusion, we have studied the magnetization properties and the groundstates of one-dimensional RFIM chains using a random walk construction for the GS structure. The finite-temperature properties, complicated by thermal excitations, are explained with the help of almost-absorbing walks. The results illustrate how global optimization influences physics at $T > 0$ *via* the geometric arrangement of domain walls. They also present an application of random walk theory to global optimization and provide a starting point for further work (*e.g.* the presence of random bonds, correlations in the fields) including an extension of the RW algorithm to 1D random field Potts chains [16]. In both the Potts and RFIM problems the relation of the GS, as given by the RW arguments, to the dynamical systems' aspects [13,17] is of interest. An extension of the random walk arguments to higher dimensional (RFIM) problems [15] seems difficult, but the basic physics should remain the same: the overlap of the GS and the low-temperature equilibrium state is finite, and can be understood in terms of the excitations being again governed by two different scales, as in 1D.

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