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Finite-size scaling analysis of exact ground states for $\pm J$ spin glass models in two dimensions

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Abstract. – With the help of *exact* ground states obtained by a polynomial algorithm we compute the domain wall energy Δ at zero temperature for the bond-random and the site-random Ising spin glass model in two dimensions. We find that in both models the stability of the ferromagnetic *and* the spin glass order ceases to exist at a *unique* concentration p_c for the ferromagnetic bonds. In the vicinity of this critical point, the size and concentration dependence of the first *and* second moment of the domain wall energy are, for both models, described by a *common* finite-size scaling form. Moreover, below this concentration the stiffness exponent turns out to be slightly negative, $\theta_S = -0.056(6)$, indicating the absence of any intermediate spin glass phase at non-zero temperature.

In spite of two decades of research on the Edwards-Anderson (EA) model for spin glasses, a number of fundamental questions failed to be answered conclusively [1], in particular those concerning the existence of a phase transition in three dimensions with [2], [3] and even without [4] an external field. In addition, the situation in two dimensions has not been clarified in the case where the ratio of ferro- and anti-ferromagnetic bonds varies. It has been argued [5]-[7] that an intermediate spin glass phase might be present in the p-T phase diagram between the ferromagnetic phase and the paramagnetic phase. In fig. 1, such a p-T phase diagram is shown including the proposed spin-glass transition line represented by the dash-dotted line. For the site-random model the evidence for the existence of a spin glass phase seems to be even stronger than for the bond-random model [8], [9].

On the other hand, in the case of the bond-random $\pm J$ model with $p = 1/2$, arguments for the absence of a spin glass phase in two dimensions were mainly based on results from Monte Carlo simulations [10], [11] and the estimates of the domain wall energy [7], [12], [13]. The data from Monte Carlo simulations, however, are not available at very low temperature. Furthermore, it is not clear whether the "stiffness" exponent θ_S is really negative [13]. More recently results of Monte Carlo simulation at lower temperatures have been reported [14] indicating a transition at $T \simeq 0.24\,\mathrm{J}$. In addition, $\theta_S = 0$ was suggested [7] based on estimates of the domain wall energy, which implies a marginally stable or weakly ordered low-temperature phase.

The aim of the present letter is to reinvestigate this issue by studying the domain wall energy at zero temperature via the determination of *exact* ground states for large system sizes and huge sample numbers. This can be done very efficiently with the help of a polynomial algorithm described by Barahona *et al.* [5], which amounts to finding a minimum-weight perfect matching in a weighted graph with $N = L^2$ nodes and has computational complexity $\mathcal{O}(N^3)$. The model that we consider is the two-dimensional Ising spin glass with binary couplings defined by the Hamiltonian

$$\mathcal{H} \equiv -\sum_{(ij)} J_{ij} S_i S_j \,, \tag{1}$$

where $S_i = \pm 1$ are Ising spins, (ij) are nearest-neighbor sites on an $L \times L$-square lattice and the interactions strengths J_{ij} are quenched random variables taking on one of the two values, $+J$ and $-J$. We consider two different cases: In the bond-random model all interactions living on the *bonds* are independently distributed with a concentration $p \in [0, 1]$ of ferromagnetic bonds ($J_{ij} = +1$). For the site-random model one first generates independently distributed random variables for all *sites*, $\epsilon_i = \pm 1$. The concentration of type-A sites, *i.e.* those with $\epsilon = +1$, is c, the type-B sites ($\epsilon = -1$) occur with probability $1 - c$. Then, J_{ij} is set to be $-J$ if and only if $\epsilon_i = \epsilon_j = -1$, and it is set to be $+J$, otherwise. In this case the ferromagnetic bond concentration is given by $p = (2 - c)c$.

We calculate the domain wall energy Δ defined by $\Delta \equiv E_{\rm p} - E_{\rm a}$, where $E_{\rm p}$ and $E_{\rm a}$ are the ground-state energies with the periodic and the anti-periodic boundary conditions in the x-direction, respectively. Free boundary conditions are imposed in the y-direction. Of crucial importance are the exponents ρ and $\theta_{\rm S}$ that characterize the system size dependence of the moments of the domain wall energy:

$$[\Delta] \propto L^{\rho} \qquad \text{and} \qquad [\Delta^2]^{1/2} \propto L^{\theta_{\rm S}} \,. \tag{2}$$

A positive value for ρ indicates the stability of a ferromagnetic ground state even in the presence of thermal fluctuations and thus the existence of the ferromagnetic long-range order at finite temperature [12]. On the other hand, a positive value for the stiffness exponent, $\theta_{\rm S}$, with ρ being negative at the same time, still indicates the stability of the ground state, which now possesses long-range order different from a ferromagnetic one. Thus a positive $\theta_{\rm S}$ is interpreted as a sign for a spin glass phase at non-zero temperature [15]. We define $p_{\rm c}^{(1)}$ and $p_{\rm c}^{(2)}$ as the critical concentrations of ferromagnetic bonds at which the asymptotic L dependences of $[\Delta]$ and $[\Delta^2]^{1/2}$, respectively, change from increasing to decreasing, *i.e.* the concentrations where a ferromagnetic phase and a spin glass phase, respectively, cease to exist at finite temperature.

We computed $[\Delta]$ and $[\Delta^2]^{1/2}$ for $L = 4, 6, 8, 12, 16, 24$ and 32 at various values of p ranging from 0.50 up to 0.95. While the number of bond samples depends on L and p, it is 32768 for one of the most time consuming data points, such as the one for $L = 32$ and $p = 0.5$. We hypothesize the following finite-size scaling form for $[\Delta]$:

$$[\Delta] L^{\psi_1} = f_1((p - p_{\rm c}^{(1)}) L^{\phi_1}) \,, \tag{3}$$

where we implicitly assume a diverging length scale proportional to $|p - p_{\rm c}|^{-\nu_1}$ (with $\nu_1 \equiv 1/\phi_1$) at a critical concentration $p_{\rm c}^{(1)}$, in analogy with a percolation transition [16]. At the critical point ($p = p_{\rm c}^{(1)}$), $[\Delta]$ diverges with system size L as $[\Delta] \propto L^{|\psi_1|}$. Its behavior above ($[\Delta] \sim L^{\rho}$) and below this concentration is contained in the asymptotic form of the scaling funtion, which is then expected to behave as $f_1(x) \sim x^{(\rho+\psi_1)/\phi_1}$ for $x \to +\infty$.

The critical exponents ϕ_1 and ψ_1 as well as $p_{\rm c}^{(1)}$ have to be chosen so that a good data collapse for all data is obtained. To quantify the "goodness" of this fit, we used an appropriate

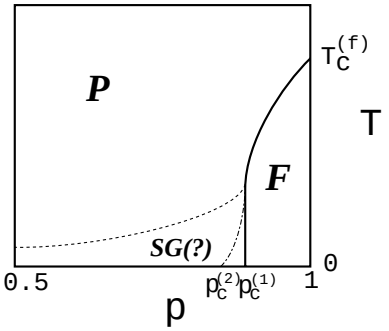

Fig. 1

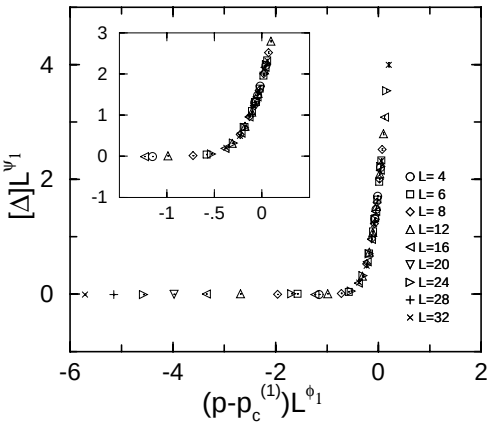

Fig. 2

Fig. 1. – The schematic phase diagram with the previously proposed spin-glass transition lines. $T_c^{(f)}$ stands for the critical temperature of the Ising model on the square lattice. P, F, and SG stand for the paramagnetic, ferromagnetic and spin glass phases, respectively.

Fig. 2. – The scaling plot of $[\Delta]$ for the bond-random model. $p_c^{(1)} = 0.896$, $\phi_1 = 0.77$ and $\psi_1 = -0.19$ are assumed. The inset is the view focused on the region near $p = p_c^{(1)}$.

cost function $S(p_c, \phi, \psi)$ [3] whose minimum value should be close to unity when the fit is statistically acceptable. When we use all the obtained data points, the best fit is achieved with $S(p_c, \phi, \psi) \sim 2$, which indicates that there is a non-negligible systematic error, *i.e.* a correction to scaling. Therefore, we have tried a similar analysis on a restricted set of data, omitting data which are presumably outside the asymptotic scaling regime, namely, data with p far from $p_c^{(1)}$ and data for small system sizes. For instance, the goodness of the fit can be significantly improved to $S = 1.12$ by omitting the data for $p = 0.95$, which yields the estimates

$$p_c^{(1)} = 0.896(1), \quad \phi_1 = 0.77(1), \quad \psi_1 = -0.19(2)\,. \tag{4}$$

Considering the very small errors accompanying the data points, it is remarkable that all the data, even including those for $L = 4$, can be expressed by the finite-size scaling form (3). The resulting scaling plot is shown in fig. 2. We have confirmed that other choices of data points producing a value of S close to unity result in estimates of $p_c^{(1)}$, ϕ_1, and ψ_1 consistent with the estimates quoted above. The value of $p_c^{(1)}$ is consistent with most of the previous estimates such as 0.88(2) [17], 0.89(2) [5] and 0.89(1) [18], while inconsistent with 0.885(1) [7].

In fig. 3, $[\Delta^2]^{1/2}$ is plotted against L. The lowest curve with crosses, which is almost straight with a negative but very small slope, corresponds to $p = 1/2$. In other words, the domain wall energy decreases systematically but it does so very slowly. The method of least squares using all the data points yields $\theta_S = -0.052(1)$, whereas the analysis with all but the first two points (for $L = 4$ and 6) results in $\theta_S = -0.060(2)$. Therefore, we quote here the value $\theta_S = -0.056(6)$ as our estimate.

Considering the size of actual reduction in $[\Delta^2]^{1/2}$ as L grows from 4 to 32, we cannot rule out the possibility that the exact value of this exponent is 0, *i.e.* $[\Delta^2]^{1/2}$ stays constant. Such a scenario would be consistent with a suggestion by Ozeki [7]. In this case one has a marginal situation and we cannot decide whether the long-range order persists at a low but finite temperature based solely on a calculation of the stiffness exponent. We may, however,

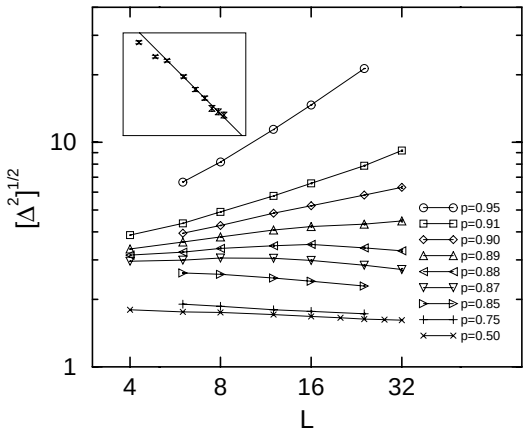

Fig. 3. – The domain wall energy $[\Delta^2]^{1/2}$ of the bond-random model plotted against the system size L for various ferromagnetic-bond concentration p. The inset is the view focused on the data points for $p = 1/2$. The straight line in the inset is obtained by the fitting to the data points excluding the two leftmost ones.

say that the low-temperature phase is only weakly ordered even if the phase transition takes place at a finite temperature.

Similarly to the above-mentioned procedure employed for $[\Delta]$, we perform a finite-size scaling analysis for the data of $[\Delta^2]^{1/2}$,

$$[\Delta^2]^{1/2} L^{\psi_2} = f_2((p - p_c^{(2)}) L^{\phi_2}), \tag{5}$$

and one expects for the scaling function $f_2(x) \sim |x|^{(\theta_S+\psi_2)/\phi_2}$ for $x \to -\infty$. Now we have to omit more data to get an acceptable fit with the value of S close to unity, indicating that the correction to scaling is larger for $[\Delta^2]^{1/2}$ than for $[\Delta]$. However, a good data collapse is obtained when we use only data for $L \geq 12$ and $0.85 \leq p \leq 0.91$. The best fit yields

$$p_c^{(2)} = 0.894(2), \quad \phi_2 = 0.79(6), \quad \psi_2 = -0.16(4) \tag{6}$$

with $S = 0.99$. The resulting scaling plot is shown in fig. 4. The present estimate of $p_c^{(2)}$ is larger than but marginally consistent with all the previous estimates such as 0.86(2) [19], 0.85 [5] and 0.870 [20], while it is clearly inconsistent with 0.854(2) [7].

It is remarkable that not only $p_c^{(2)}$ but also ϕ_2 and ψ_2 agree with the corresponding values in (4) within the statistical errors. While the agreement in p_c already suggests the absence of the intermediate phase, we consider the agreement in the critical indices as another evidence for the absence of the intermediate spin glass phase, since it is hardly possible that the first and the second moment of Δ show the same critical behavior at different values of p_c.

We now focus on the site-random model. In fig. 5, $[\Delta^2]^{1/2}$ is plotted as a function of the system size. A significant correction to scaling can be seen in fig. 5. The same remark applies to the first moment, $[\Delta]$. We have performed a finite-size scaling analysis similar to what has been done for the bond-random model. As for $[\Delta]$, in order to reduce the cost function, S, down to unity, the smallest system sizes $L = 4$ and $L = 6$ have to be excluded from the scaling plot. The data out of the range, $0.60 \leq c \leq 0.68$, are also excluded in the quantitative estimation of c_c and the indices ϕ_1 and ψ_1. As for $[\Delta^2]^{1/2}$, an even stronger correction to scaling is present, as shown in fig. 5, making an additional system size ($L = 8$) unavailable for the quantitative estimation of c_c. The range of c from which the data are chosen is again $0.60 \leq c \leq 0.68$.

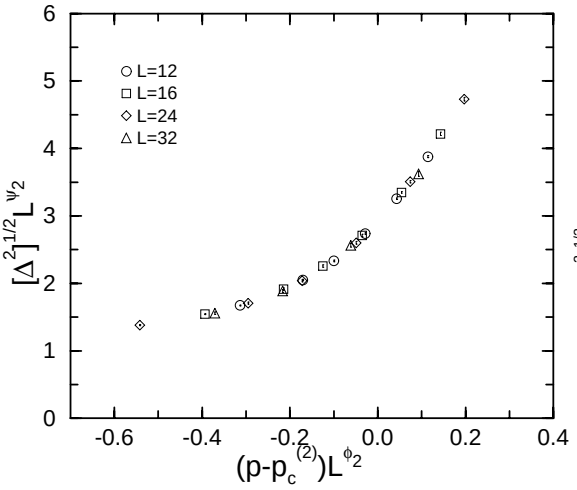

Fig. 4

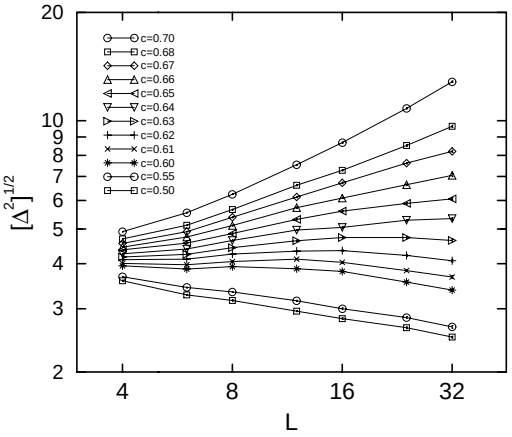

Fig. 5

Fig. 4. – The scaling plot of $[\Delta^2]^{1/2}$ of the bond-random model. $p_c^{(2)} = 0.894$, $\phi_2 = 0.79$ and $\psi_2 = -0.16$ are assumed.

Fig. 5. – The domain wall energy $[\Delta^2]^{1/2}$ of the site-random model plotted against the system size L for various "A-site" concentration c.

The critical concentration, $c_c^{(i)}$, and the critical indices, ϕ_i and ψ_i, are defined in a similar fashion to (3) and (5), resulting in

$$c_c^{(1)} = 0.658(3), \quad \phi_1 = 0.78(2), \quad \psi_1 = -0.18(3)\,, \tag{7}$$

$$c_c^{(2)} = 0.661(4), \quad \phi_2 = 0.79(2), \quad \psi_2 = -0.23(3)\,. \tag{8}$$

Obviously $c_c^{(1)} = c_c^{(2)}$ within the error bars, from which also we here conclude that no intermediate spin glass phase exists. In addition, again, the critical indices for $[\Delta]$ agree with those for $[\Delta^2]^{1/2}$ within the statistical errors. These critical indices agree with those for the bond-random model ((4) and (6)), implying that *both models belong to the same universality class.*

To summarize, we have performed a systematic calculation of the domain wall energy at zero temperature with systems larger than previous calculations for both bond-random and site-random models. We observed a significant crossover effect or a correction-to-scaling especially for the site-random model, while no indication for a finite temperature spin glass phase could be detected. Some of the previous evidences for the positive stiffness exponents were based on systems smaller than those studied in the present paper and, therefore, may be attributed to this crossover effect. The critical concentration for the ferromagnetic bonds and critical indices estimated from $[\Delta]$ agree with those from $[\Delta^2]^{1/2}$, again indicating the absence of an intermediate phase. Moreover, the critical exponents for the site-random model agree with those for the bond-random model, suggesting that both models have the same universal critical behavior and also qualitatively identical features away from p_c. We have also seen that the domain wall energy is almost independent of the system size below p_c.

It is interesting to speculate about some sort of "unfrustrated" clusters and regard the transition that we have observed at $p = p_c$ as a percolation transition of such clusters [21]. The correlation length exponent for two-dimensional percolation [16] is $\nu = 4/3$, which is indeed compatible with our estimate $\nu = 1/\phi \approx 1.3$. This agreement makes it worth putting some more effort in the investigation of this issue.

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