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Average persistence of random walks

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Abstract. – We study the first passage time properties of an integrated Brownian curve both in homogeneous and disordered environments. In a disordered medium we relate the scaling properties of this center-of-mass persistence of a random walker to the average persistence, the latter being the probability $\overline{P}_{\rm pr}(t)$ that the expectation value $\langle x(t)\rangle$ of the walker's position after time t has not returned to the initial value. The average persistence is then connected to the statistics of extreme events of homogeneous random walks which can be computed exactly for moderate system sizes. As a result we obtain a logarithmic dependence $\overline{P}_{\rm pr}(t) \sim \ln(t)^{-\overline{\theta}}$ with a new exponent $\overline{\theta} = 0.191 \pm 0.002$. We note a complete correspondence between the average persistence of random walks and the magnetization autocorrelation function of the transverse-field Ising chain, in the homogeneous and disordered case.

First passage time or persistence problems have a long history in the physical literature [1]. Recently they gained a lot of interest, since the persistence exponents, describing the asymptotic behavior of first passage time probabilities, are shown to be independent dynamical critical exponents, which have been calculated for various models exactly [2, 3]. Not much is known about analogous quantities in systems with quenched disorder, for instance the random walk (or diffusion) in a disordered environment [4], which is in the one-dimensional case the Sinai-model [5]. For this model the first passage / persistence exponent for a single walker has been determined by us in a previous work [6]. In this letter we introduce and study the concept of *average* persistence of random walks both in homogeneous and random environments.

We consider a random walk with nearest-neighbor hopping in one dimension defined by the Master equation

$$p_i(t=0) = \delta_{i,1}\ , \quad \frac{\mathrm{d}}{\mathrm{d}t}p_i(t) = -(w_{i,i-1} + w_{i,i+1})p_i(t) + w_{i-1,i}p_{i-1}(t) + w_{i+1,i}p_{i+1}(t) \qquad (1)$$

describing the time evolution of the probability $p_i(t)$ for the walker to be at site i after time t when having been initially at site $i = 1$. The homogeneous random walk is defined via uniform transition rates $w_{i,i+1} = w_{i+1,i} = 1/2$ and the random walk in a disordered environment is modeled by choosing the transition rates to be quenched random variables that obey a particular distribution, *e.g.*, the uniform distribution $\pi_{\rm uni}$ given by $\pi_{\rm uni}(w_{i,i\pm1}) = 1$ for $0 < w_{i,i\pm1} < 1$ and 0 otherwise, or the binary distribution

$$\pi_{\rm bin}(w_{i,i-1}) = \frac{1}{2}\delta(w_{i,i-1}-\lambda) + \frac{1}{2}\delta(w_{i,i-1}-\lambda^{-1}), \qquad w_{i,i+1} = 1, \quad \forall\, i \tag{2}$$

with λ some arbitrary parameter. For the disordered case physical observables have to be averaged over this distribution π which is denoted by square brackets $[\ldots]_{\rm av}$. Note we consider the general case of asymmetric hopping rates $w_{i,i+1} \neq w_{i+1,i}$ and we do not confine ourselves to the so-called random force model with correlated transition probabilities parameterized as $w_{i,i\pm1} = A\exp[\pm\phi_i]$ with random, uncorrelated potentials ϕ_i on each site.

In order to define *single-walker* persistence probabilities we put an adsorbing boundary at site $i = 0$, which means that we set $w_{0,1} = 0$ and also introduce a finite-size length scale L into the system by putting another adsorbing boundary at $i = L+1$, *i.e.* setting $w_{L+1,L} = 0$. Now we define the length-scale–dependent single-walker persistence $P_{\rm pr}(L,t)$ to be the probability that a walker does not cross its starting point (*i.e.* does not get trapped at site $i = 0$) within the time interval t. The following arguments [6] will then lead us to a scaling form for $P_{\rm pr}(L,t)$: a) the typical time the walker needs to reach the site $L+1$ scales like $t_{\rm typ} \sim L^2$ in the case with symmetric transition rates and like $\ln t_{\rm typ} \sim \sqrt{L}$ in the asymmetric case, the Sinai model, b) the asymptotic limit $P_{\rm surv}(L) = \lim_{t\to\infty} P_{\rm pr}(t,L) = \lim_{t\to\infty} p_{L+1}(t)$, which is what we call a survival probability, behaves like $P_{\rm surv}(L) \sim L^{-1}$ in the symmetric case and like $P_{\rm surv}(L) \sim L^{-\theta_{\rm a}}$ with $\theta = 1/2$ in the Sinai model. Hence we expect

$$[P_{\rm pr}(L,t)]_{\rm av} \sim \begin{cases} L^{-\theta_{\rm s}} \cdot \tilde{p}_{\rm s}(L^2/t) & \text{(symmetric case)}\,, \\[2ex] L^{-\theta_{\rm a}} \cdot \tilde{p}_{\rm a}(\sqrt{L}/\ln t) & \text{(asymmetric case)}\,, \end{cases} \tag{3}$$

where the persistence exponents are $\theta_{\rm s} = 1$ for the symmetric case and $\theta_{\rm a} = 1/2$ for the asymmetric case, and the scaling functions behave like $\tilde{p}_{\rm s}(x) \to x^{\theta_{\rm s}/2}$ and $\tilde{p}_{\rm a}(x) \to x^{2\theta_{\rm a}}$ for $x \to \infty$ and $\tilde{p}_{\rm s/a}(x) \to$ const for $x \to 0$. In the infinite-system-size limit one thus has for persistence probability $P_{\rm pr}(t)$ in the asymmetric case $P_{\rm pr}(t) = \lim_{L\to\infty} P_{\rm pr}(L,t) \propto \ln(t)^{-2\theta_{\rm a}}$, a logarithmically slow decay reminiscent of the critical dynamics of the surface magnetization of the random transverse Ising chain [7] (RTIC). This is not incidental: the equivalence of the surface magnetization in the latter quantum spin chain and the survival probabilities of random walks has been formulated the first time in [8] and further analogies between anomalous diffusion and the RTIC have been uncovered recently [6, 9].

It is known that diffusion in the Sinai model is different from normal diffusion in many respects [4]. Consider for instance one particular disorder realization. Then an initially narrow probability distribution of a walker peaked around, say, $x(t=0) = 0$ does not broaden with time, only its expectation value $\langle x(t)\rangle = \sum_i i \cdot p_i(t)$ diffuses logarithmically slowly away from its starting point. Therefore, in this situation in addition to the single-walker persistence one should also consider the persistence properties of the average position of the walker $\langle x(t)\rangle$, the *average* persistence $\overline{P}_{\rm pr}(t)$: This is the probability that up to a specified time t the average position of the walk in a particular environment has always been on one side of the starting point, *i.e.* $\forall\, 0 < t' < t:\ \langle x(t')\rangle > 0$.

To study the average persistence we consider again a finite-size situation where we put an adsorbing boundary at $i = -L$ and one at $i = L$ (note that now a single walker is not adsorbed when crossing the starting point but only when he leaves the finite strip of width $2L$ centered around $i = 0$). Then the average persistence $\overline{P}_{\rm pr}(L,t)$ obeys the same scaling form as in the second line of (3), but the single-walker persistence exponent $\theta_{\rm a}$ is replaced by the average

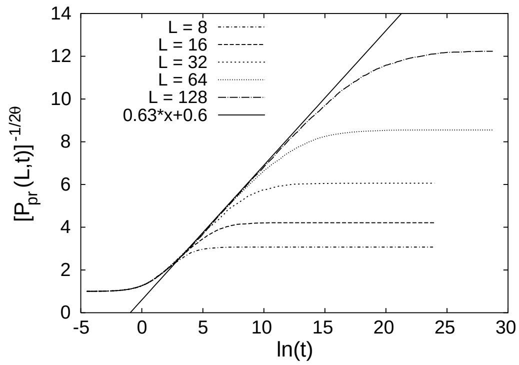

Fig. 1. – The average persistence $[\overline{P}_{\rm pr}(L,t)]_{\rm av}$ to the power $-1/2\overline{\theta}_{\rm a}$ *vs.* $\ln t$, which should yield a straight line in the limit $L \to \infty$, for different finite-size scaling length L. The data are for uniform distribution and averaged over 50000 samples. We used an estimate $\overline{\theta}_{\rm a} = 0.19$ to obtain an optimal data collapse in the scaling region. The straight line indicates the asymptotic result for the infinite system $L \to \infty$.

persistence exponent $\overline{\theta}_{\rm a}$:

$$[\overline{P}_{\rm pr}(L,t)]_{\rm av} \sim L^{-\overline{\theta}_{\rm a}} \cdot \overline{p}_{\rm a}(\sqrt{L}/\ln t)\ . \tag{4}$$

Moreover, the persistence probability for the infinite system decays again logarithmically slow: $\overline{P}_{\rm pr}(t) \propto \ln(t)^{-2\overline{\theta}_{\rm a}}$. Recently it has been conjectured for the random force model [9] that the exponent $\overline{\theta}_{\rm a}$ is related to the golden mean via $\overline{\theta}_{\rm a} = (3-\sqrt{5})/4 \approx 0.191$. In fig. 1 we show numerical data that have been obtained by a numerical calculation of $P_{\rm pr}(L,t)$ via diagonalization of the linear operator on the r.h.s. of eq. (1), which indicate that this might also hold for the general asymmetric case.

In what follows we will demonstrate how to obtain a precise estimate of the average persistence exponent $\overline{\theta}_{\rm a}$. Since in the limit $t \to \infty$ the average persistence probability is given by $\overline{P}_{\rm surv} \sim L^{-\overline{\theta}_{\rm a}}$, the computation of the survival probability $\overline{P}_{\rm surv}$ will lead us to the desired result. We will now relate this survival probability to a problem in the statistics of

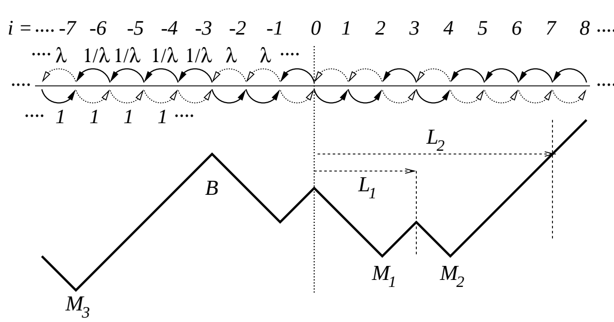

Fig. 2. – Sketch of the random landscape that corresponds to the binary distribution (2). In the limit $\lambda \to 0$ transitions corresponding to broken arrows will rarely occur, whereas transitions corresponding to full arrows will occur with probability close to one. For further details describing the effective dynamics see text.

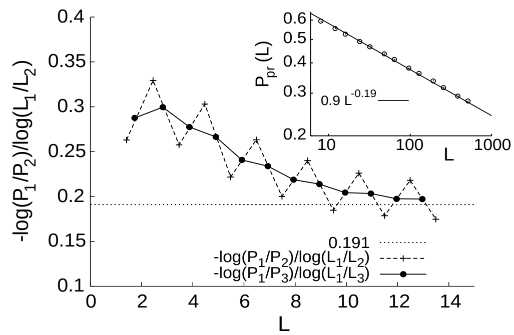

Fig. 3. – The effective finite-size exponent $\overline{\theta}_{\mathrm{a}}$ from exact values for $\overline{P}_{\mathrm{pr}}(L)$ via complete enumeration of all surviving configurations. The asymptotic value $\overline{\theta}_{\mathrm{a}} = 0.191$ obtained by series extrapolation methods is also shown. The insert shows the result for $\overline{P}_{\mathrm{pr}}(L)$ via stochastic enumeration for larger system sizes, giving $\overline{\theta}_{\mathrm{a}} = 0.19 \pm 0.01$.

extreme events of homogeneous random walks. To this end we consider the binary distribution π in (2) in the limit $\lambda \to 0$, which means that one can discriminate between "forward" bonds $(i, i+1)$, which are those for which $w_{i,i-1} = \lambda \to 0$, *i.e.* those where almost only (with probability $1/(1+\lambda) \to 1$) jumps from i to $i+1$ occur, and "backward" bonds, which have $w_{i,i-1} = \lambda^{-1} \to \infty$, implying almost always jumps from i to $i-1$ (again with probability $\lambda^{-1}/(1+\lambda^{-1}) \to 1$). This implies that when sketching the transition rates as configuration as is done in fig. 2 the disorder configuration can be done visualized as a random landscape with hills and valleys. Thus a walker starting at $i = 0$ is *on average* first driven to the first minimum M_1, and considering the finite-size situation where $L = L_1$ this disorder configuration counts as a surviving configuration, since the walker in average spends most of its time at M_1. Increasing the length scale L further beyond L_2 will then drive the average position of the walker over the barrier B into the valley M_3 at *negative* coordinates, which means that this configuration is dead, *i.e.* not surviving.

Thus we conclude that the computation of $\overline{P}_{\mathrm{surv}}$ amounts to counting all surviving configurations of a random landscape, *i.e.* a homogeneous random walk, in the manner described above and the ratio of surviving configurations is just $\overline{P}_{\mathrm{surv}}$. In other words we consider random walks of length $2L$, corresponding to realizations of the transition probabilities according to (2) in a strip of width $2L$ centered around the starting point, and say that it is on average surviving if certain conditions are fulfilled. These conditions are checked by inspecting the random landscape generated by the disorder configuration (*i.e.* the transition rates): one scans the landscape in both directions from the starting point and denotes with $h(i)$ $(i = -L, -L+1, \ldots, -1, 0, 1, 2, \ldots, L)$ the position of the walker (or height of the landscape) at step (or site) i. We define the extreme events to the right and to the left of $i = 0$ by $x_{\mathrm{max/min}}(i) = \mathrm{max/min}\{h(j)|0 \le j \le i\}$ and $y_{\mathrm{max/min}}(i) = \mathrm{max/min}\{h(j)| -i \le j \le 0\}$, respectively, and check iteratively (from $i = 1$ to $i = L$) whether

$$x_{\mathrm{min}}(i) > y_{\mathrm{min}}(i) \quad \text{and} \quad x_{\mathrm{max}}(i) > y_{\mathrm{max}}(i) \,. \tag{5}$$

If this happens for some site i, as it does for $i = 8$ in fig. 2, it means that there is a lower minimum on the left side of the starting point ($x_{\mathrm{min}} < y_{\mathrm{min}}$) and that the walker can go there since the barrier in between is low enough ($x_{\mathrm{max}} < y_{\mathrm{max}}$). This implies that this configuration is dead.

In the inset of fig. 3 we show the results for a numerical estimate of the survival probability

$\overline{P}_{\rm surv}(L)$ by inspecting 10^5 random walk configuration for different system sizes —the data fit well to a power law with the exponent $\overline{\theta}_{\rm a} = 0.19$. Next we implemented a recursive routine that computes the number of surviving configuration according to the above extreme events criterion (5) *exactly*. Here we took special care to the degenerate minima (like M_1 and M_2 in fig. 2), in which case we assumed the arithmetic mean of the location of the two to be the effective position (L_1 in fig. 2). From these exact data for system sizes up to $L = 14$ we can extract via $\overline{\theta}_{\rm a}(L_a, L_b) = -\ln(P_{\rm surv}(L_a)/P_{\rm surv}(L_b))/\ln(L_a/L_b)$ an effective finite-size exponent that approaches the exact value. In fig. 3 we show these exact data for $L_a = L = L_b + 1$ and $L_a = L = L_b + 2$. Finally, from a standard series extrapolation procedure [10] applied to these data we obtain our best estimate which is

$$\overline{\theta}_{\rm a} = 0.191 \pm 0.002\,, \tag{6}$$

in very good agreement with the conjectured value $\overline{\theta}_{\rm a} = (3 - \sqrt{5})/4 = 0.19098\ldots$ for the random force model [9].

Although the concept of an average persistence seems to be based upon the special feature of non-dispersive diffusion in the Sinai model, which we described above, it turns out that there is an analogy to it in normal diffusion, too. To show this, we first introduce the integrated position of the walker at step t as $I_t = \sum_{\tau=1}^{t} i_\tau$, where i_τ is its position at time-step τ. Then, we define the *center-of-mass persistence* through the survival condition for the integrated position as $I_{t'} \geq 0$ for $0 < t' < t$. For the disordered case we define a center-of-mass position via

$$C(i) = \sum_{j=-i}^{i} j \cdot M(h(j))\,, \tag{7}$$

where the weights $M(h(j))$, which are proportional to the average time the walker spent at a given site, depend on the random configurations. In a strip ($-L \leq i \leq L$) with adsorbing boundaries the survival condition of the average position in the large t-limit is $C(i) \geq 0$ for $i = 1, \ldots, L$, which is equivalent to the survival condition in (5). For the extreme binary distribution we consider here, the weight function is singular, such that $M(h')/M(h) \to 0$ for $h' < h$, which is a consequence of the limit $\lambda \to 0$ for the distribution (2). Since the form of the distribution of the random transition rates is generally irrelevant one expects that the scaling properties of the average persistence and those of the center-of-mass persistence in (7) are equivalent for other types of distributions as well.

For a homogeneous walk one can also study the center-of-mass persistence or the survival probability of the I_t integrated position. This type of problem has already been considered by Sinai [11] for a discrete model and later by Burkhardt [12] for the continuum model. According to these exact results in a homogeneous infinite system the long-time behavior of the average persistence is given by $\lim_{L\to\infty} \overline{P}_{\rm pr}^{\rm hom}(L,t) \sim t^{-1/4}$. In the following we generalize this result involving a length scale, L, which can be done in two different ways. If we consider the walk in finite a strip ($-L \leq i \leq L$) with adsorbing boundaries, thus the walk is being adsorbed if its *position* $i > L$, then we have

$$\overline{P}_{\rm pr}^{\rm hom(pos)}(L,t) = t^{-\overline{\theta}_{\rm hom}} \overline{p}_{\rm hom}^{\rm (pos)}(L^2/t)\,. \tag{8}$$

Here $\overline{\theta}_{\rm hom} = 1/4$, the scaling function behaves like $\overline{p}_{\rm hom}^{\rm (pos)}(x) \sim x^{\overline{\theta}_{\rm hom}}$ for $x \to 0$ and $\overline{p}_{\rm hom}^{\rm (pos)}(x) \to$ const for $x \to \infty$, and the scaling combination, $x = L^2/t$ in (8), follows from the scaling properties of the Brownian motion.

Another condition for the length scale, L, can be formulated with the integrated position, I_t, so that the walker is adsorbed if $I_t > L$. Now, to obtain the scaling relation between L

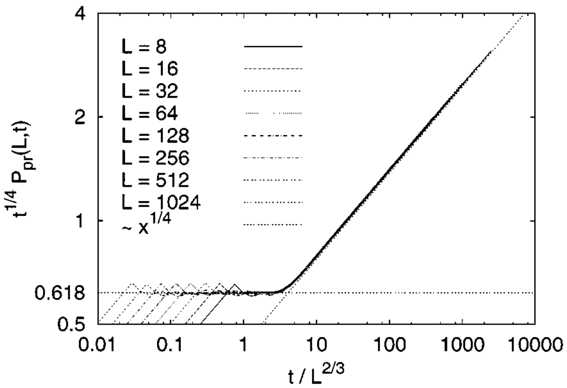

Fig. 4. – Finite size scaling plot for the average persistence probability $\overline{P}_{\rm pr}^{\rm hom}(L,t)$ for the homogeneous random walk (9), *i.e.* the adsorbing condition is on the intergated position: $I_t > L$. The data are obtained generating 10^5 independent random walks with 10^4 steps each, the straight lines being the asymptotic forms of the scaling function $\overline{p}_{\rm hom}(x)$ for $x \to \infty$ and for $x \to 0$ (see text).

and t we make use the fact that $\langle I_t^2 \rangle = \sum_{\tau=1}^{t} \tau^2 = t(t+1)(2t+1)/6 \sim t^3$, thus $t \sim L^{2/3}$. With this we have for the persistence probability with adsorbing condition for the integrated displacement

$$\overline{P}_{\rm pr}^{\rm hom}(L,t) = t^{-\overline{\theta}_{\rm hom}} \overline{p}_{\rm hom}(L^{2/3}/t)\,, \tag{9}$$

where, as in (8), $\overline{\theta}_{\rm hom} = 1/4$. The scaling function $\overline{p}_{\rm hom}(x)$ has the same limiting behaviour as $\overline{p}_{\rm hom}^{(\rm pos)}(x)$ in (8).

We checked the scaling relations in (8) and (9) numerically and show the result for $\overline{P}_{\rm pr}^{\rm hom}$ (*i.e.* adsorbing condition for the integrated position) in fig. 4. By an exact enumeration up to $t = 30$ we have also estimated $\lim_{x\to\infty} \overline{p}_{\rm hom}(x) = 0.6183(3)$ for the scaling function, whose

TABLE I. – *Comparison of the scaling behavior of the average* (*single-walker*) *persistence of a random walk* (RW), $P_{\rm pr}(L,t)$, *with the bulk* (*surface*) *spin-spin autocorrelation function of the transverse-field Ising model* (TIM), $G(L,t)$. *Both homogeneous and random systems are considered.*

	RW	TIM
	$P_{\rm pr}(L,t)$	$G(L,t)$
	average position	bulk spin ($\sigma_{\rm b}$)
Homogeneous	$t^{-1/4}\overline{p}_{\rm h}(L^{2/3}/t)$	$t^{-1/4} g_{\rm h}^{\rm b}(L/t)$ [13]
Random	$L^{-0.191}\overline{p}_{\rm r}(\sqrt{L}/\ln t)$	$L^{-x_{\rm r}} g_{\rm r}^{\rm b}(\sqrt{L}/\ln t)$ [7]
		$x_{\rm r} = (3-\sqrt{5})/4 = 0.191$ [14]
	single walker	surface spin ($\sigma_{\rm s}$)
Homogeneous	$L^{-1} p_{\rm h}(L^2/t)$	$L^{-1} g_{\rm h}^{\rm s}(L/t)$ [13]
Random	$L^{-1/2} p_{\rm r}(\sqrt{L}/\ln t)$	$L^{-1/2} g_{\rm r}^{\rm s}(\sqrt{L}/\ln t)$ [7]

value agrees, within the accuracy of the estimate, with the inverse of the golden mean ratio $1/\tau = 2/(\sqrt{5}+1) = 0.6180$.

Concluding, we introduced the concept of average persistence in random walks and studied its scaling properties in homogeneous and in disordered environments. The scaling behavior of $\overline{P}_{\rm pr}(L,t)$ as given in (4) and (8), (9) involve new exponents: $\overline{\theta}$ in (6) and $\overline{\theta}_{\rm hom} = 1/4$ for the disordered and homogeneous problems, respectively. It is interesting to note an analogy between the scaling form of the average persistence of random walks and that of the magnetization autocorrelation function $G_{\rm b,s}(L,t) = [\langle\sigma^x_{\rm b,s}(0)\sigma^x_{\rm b,s}(t)\rangle]_{\rm av}$ of the transverse-field Ising spin chain of length L. As summarized table I these relations complete the previously observed correspondences [6] between single-walker persistence and the surface autocorrelation function of the transverse-field Ising spin chain.

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REFERENCES

[1] van Kampen N. G., *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, New York) 1981.

[2] Derrida B. and Pomeau Y., *Phys. Rev. Lett.*, **48** (1982) 627; Derrida B., Bray A. J. and Godrèche C., *J. Phys. A*, **27** (1994) L357; Derrida B., Hakim V. and Pasquier V., *Phys. Rev. Lett.*, **75** (1995) 751.

[3] Majumdar S. N. and Sire C., *Phys. Rev. Lett.*, **77** (1996) 1420; Majumdar S. N., Sire C., Bray A. J. and Cornell S. J., *Phys. Rev. Lett.*, **77** (1996) 2867; Majumdar S. N., Bray A. J., Cornell S. J. and Sire C., *Phys. Rev. Lett.*, **77** (1996) 3704; Oerding K., Cornell S. J. and Bray A., *Phys. Rev. E*, **56** (1997) R25; Krug J., Kallabis H., Majumdar S. N., Cornell S. J., Bray A. J. and Sire C., *Phys. Rev. E*, **56** (1997) 2702; Menyhárd N. and Ódor G., *J. Phys. A*, **30** (1997) 8515.

[4] For a review on random walks in disordered media see: Bouchaud J. P. and Georges A., *Phys. Rep.*, **195** (1990) 127.

[5] Sinai Ya. G., *Theor. Probab. Appl.*, **27** (1982) 247.

[6] Iglói F. and Rieger H., *Phys. Rev. E*, **58** (1998) 4238.

[7] Rieger H. and Iglói F., *Europhys. Lett.*, **39** (1997) 135.

[8] Iglói F. and Rieger H., *Phys. Rev. B*, **57** (1998) 11404.

[9] Fisher D. S., Le Doussal P. and Monthus C., *Phys. Rev. Lett.*, **80** (1998) 3539.

[10] Bulirsch R. and Stoer J., *Num. Math.*, **6** (1964) 413.

[11] Sinai Ya. G., *Theor. Math. Phys.*, **90** (1992) 239.

[12] Burkhardt T. W., *J. Phys. A*, **26** (1993) L1157.

[13] McCoy B. M. and Wu T. T., *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, Mass.) 1973.

[14] Fisher D. S., *Phys. Rev. Lett.*, **69** (1992) 534; *Phys. Rev. B*, **51** (1995) 6411; Crisanti A. and Rieger H., *J. Stat. Phys.*, **77** (1994) 1087; Young A. P. and Rieger H., *Phys. Rev. B*, **53** (1996) 8486.