

Disordered p -spin interaction models on Husimi trees

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Ising spin models with multisite interactions are investigated on pure Husimi trees, which is an analog to the Bethe approximation for these models in finite dimensions. The interaction strengths are quenched random variables and mean-field theory predicts a spin-glass transition from a paramagnetic phase to a spin-glass phase with nonvanishing Edwards-Anderson order parameter. We investigate different kinds of distribution for the interactions at zero temperature as well as at nonvanishing temperature and the effect of external fields. We show that on the Husimi tree the mean-field scenario does not take place for small coordination numbers, not even at zero temperature, although frustration is present after fixing the boundary spins. This result concurs with the observations made by looking at several random multisite interaction models in two-dimensional lattices. In the limit of infinite coordination number the (replica-symmetric) mean-field result is recovered.

I. INTRODUCTION

Since Baxter's solution of the eight-vertex model,¹ Ising models with multisite interaction have gained increasing interest. Several of them could be solved exactly in two dimensions,² and in three dimensions they show unusual features, such as first-order phase transitions even within an external field.³ The investigation of mean-field (or infinite-range) models with p -spin interactions that are quenched random variables led to the discovery of the only nontrivial spin-glass model that is exactly solvable in the thermodynamic limit: the random energy model, which is the $p \rightarrow \infty$ limit of the p -spin interaction spin glass.⁴

In addition, these spin-glass models show an interesting phase transition scenario that is different from the usual ($p=2$) Sherrington-Kirkpatrick (SK) model. The Edwards-Anderson (EA) order parameter jumps discontinuously to a nonvanishing value when crossing a temperature T'_g from above. Nevertheless, thermodynamic quantities are continuous since they involve only integrals over the Parisi order-parameter functions $q(x)$.⁵ Furthermore, it was possible to show that this equilibrium phase transition is preceded by a dynamical freezing transition at a temperature $T_g > T'_g$, where the correlations do not decay to zero on finite time scales.⁶ There is a close analogy to structural glass transitions,⁷ and indeed the mathematical structure of the effective equations describing the dynamical behavior of the spin-autocorrelation function are similar to those considered in mode coupling theories of the structural glass transition.⁸

So far much is known about the infinite-range version of these multisite interaction spin models with quenched disorder, but nothing about their counterparts in finite dimensions, especially two or three. In another publication we addressed the question of how to achieve a nontrivial realization of these models on regular lattices.⁹ In this paper we consider models with random p -spin interactions among spins on a pure Husimi tree,¹⁰ which is a natural generalization of the Bethe approximation for

usual two-spin interaction models to the models with multisite interactions. As in the former case, one expects this approximation to be a useful alternative to the mean-field approximation, if one is interested in the behavior of finite dimensional systems with short-range interactions.¹¹⁻¹⁵ Indeed it is found that, in contrast to the spin glass on the Bethe lattice, the Husimi tree does not have a spin-glass transition as long as the coordination number is small—reminiscent of the absence of such a transition in several two-dimensional random multisite interaction models.⁹

The outline of the paper is as follows. First we define the construction of the Husimi tree and derive recursion relations for the random fields acting upon the spins after performing a partial trace over the outer branches attached to it. Then we consider several special cases with small coordination numbers and show that a spin-glass transition is absent. In the succeeding section we demonstrate how the mean-field scenario is recovered in the limit of infinite coordination number. Finally we discuss our results, put them into a more general context—especially with respect to Potts glasses—and address some open questions.

II. THE MODEL AND RECURSION RELATIONS

A pure Husimi tree is constructed in the following way: We start with a polygon of p vertices, with one Ising spin located at each vertex. This is the first-generation branch and one of the spins is called the base spin σ_1 of the first-generation branch. All p spins interact via a single p -spin interaction J , its contribution to the Hamiltonian of the system is $-J \prod_{i=1}^p \sigma_i$. Figure 1 gives an example for $p=3$.

In the $(n+1)$ -th step one draws a new base polygon with base spin σ_{n+1} and attaches to each of the remaining $p-1$ vertices $q-1$ n th-generation branches—as is shown in Fig. 1 for $q=4$. The number q is the coordination number of the tree, in analogy to the Cayley tree. Each of the $(p-1)(q-1)$ n th generation branches has its

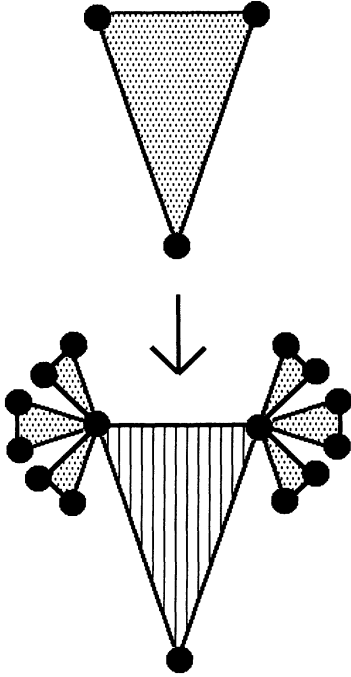


FIG. 1. Construction of the Husimi tree (Ref. 10) for $p=3$, $q=4$. The upper triangle is the first-generation branch with the base spin at the bottom. The lower part is the second-generation branch, again with the new base spin at the bottom. Since the polygons are triangles, the resulting Husimi tree is also called a cactus (Ref. 10).

own p -spin interaction strength if we deal with the disordered case. The p -spin interaction strength of the base polygon is a random variable, uncorrelated with the others. The probability distribution of the couplings is $\rho(J)$. One continues analogously up to the n_{\max} th generation and completes the graph by finally connecting together q otherwise uncorrelated n_{\max} th-generation branches at their base spins.

In this way—apart from the boundary spins—each

inner spin is placed simultaneously on the vertex of q different p -gons. Therefore $p=4$, $q=4$ should be an approximation for the square lattice with four-spin interactions among spins on the basic plaquettes; $p=3$, $q=6$ corresponds to an approximation to the triangle lattice with three-spin interaction among the spins on basic triangles; and finally in the three-dimensional fcc lattice with four-spin interactions among the spins on each elementary tetrahedron, we have $p=4$ and $q=8$.

Note that the ratio of the number of boundary spins N_n^{BS} to the number of inner spins N_n^{IS} at the n th-generation branch is always greater than 1 for all numbers n . This situation is the same in the case of a Cayley tree. The Hamiltonian of the Husimi tree is

$$\mathcal{H} = - \sum_{\text{pol}} J_{\text{pol}} \prod_{i=1}^p \sigma_{\text{pol}}^i + H \sum_i \sigma_i, \quad (1)$$

where the first sum is over all polygons of the tree and the second sum is over all spins of the tree.

Fixed boundary conditions lead generically to frustration if the p -spin interactions can take on both signs. We only consider symmetric distributions of the couplings $\rho(J)=\rho(-J)$ and after a gauge transformation we are left with only non-negative couplings. In contrast to the Bethe lattice there is some arbitrariness in the gauge transformation: Starting with the boundary polygons one can absorb the sign of their interactions by flipping one of their $p-1$ boundary spins. A boundary configuration of all spins up does not lead to complete random boundary conditions since only a fraction of $\frac{1}{2}(p-1)$ of the spins are flipped after the gauge transformation. Therefore we start with $\frac{1}{2}$ of the spins up, the others down, which is not altered after the gauge transformation. The resulting distribution of interactions is then $\rho_g(J)=2\rho(J)\theta(J)$.

Now let us connect the probability distribution $P(\sigma_{n+1})$ for the base spin of a $(n+1)$ -th-generation branch to the $(p-1)(q-1)$ probability distributions $P_{i,j}(\sigma_n^i)$ ($i=1, \dots, p-1$, $j=1, \dots, q-1$) n th-generation branches that are attached to the base polygon. This probability distribution is given by

$$P^{(n+1)}(\sigma_{n+1}) = \frac{1}{N_p} e^{\beta H \sigma_{n+1}} \sum_{\sigma_n^i} \exp \left[\beta J_{n+1} \sigma_{n+1} \prod_{i=1}^{p-1} \sigma_n^i \right] \prod_{i=1}^{p-1} \prod_{j=1}^{q-1} P_{i,j}^n(\sigma_n^i). \quad (2)$$

N_p is a normalization constant [$N_p = P(+)+P(-)$], J_{n+1} the p -spin interaction strength of the base polygon of the $(n+1)$ -th branch, and β is the inverse temperature. Note that in the isotropic case—which was investigated recently¹⁵ for $p=4$, $q=4$ —all $P_{i,j}$ are the same (starting with homogeneous initial conditions for the distribution of boundary spins), but in the disordered case they were random variables, since they depend on the specific realization of the random p -spin interactions within the corresponding branches as well as on their random boundary conditions.

The magnetization of the base spin σ_{n+1} is now given by

$$M_{n+1} = \frac{e^{\beta H} - e^{-\beta H} x_{n+1}}{e^{\beta H} + e^{-\beta H} x_{n+1}}, \quad x_{n+1} = e^{2\beta H} \frac{P^{(n+1)}(-)}{P^{(n+1)}(+)}. \quad (3)$$

In analogy to the definition of x_{n+1} one can define this quantity for the $(p-1)(q-1)$ distributions of the base spins of the n th-generation branches

$$x_n^{i,j} = e^{2\beta H} \frac{P_{i,j}(-)}{P_{i,j}(+)}. \quad (4)$$

(where we dropped the index n on the right-hand side); then one gets the relation

$$x_{n+1} = \frac{\sum_{i=1}^{p-1} \sum_{\sigma_i} \exp \left[-\beta J_{n+1} \prod_{i=1}^{p-1} \sigma_i + \beta H \sum_{i=1}^{p-1} \sigma_i \right] \prod_{i=1}^{p-1} \prod_{j=1}^{q-1} \frac{P_{ij}(\sigma_i) e^{-\beta H \sigma_i}}{P_{i,j}(+) e^{-\beta H}}}{\sum_{i=1}^{p-1} \sum_{\sigma_i} \exp \left[+\beta J_{n+1} \prod_{i=1}^{p-1} \sigma_i + \beta H \sum_{i=1}^{p-1} \sigma_i \right] \prod_{i=1}^{p-1} \prod_{j=1}^{q-1} \frac{P_{ij}(\sigma_i) e^{-\beta H \sigma_i}}{P_{i,j}(+) e^{-\beta H}}} . \quad (5)$$

It turns out that the quantity x can be identified with an effective field by

$$\beta h = -\frac{1}{2} \ln x , \quad (6)$$

so that

$$\frac{P_{i,j}(\sigma_i) e^{-\beta H \sigma_i}}{P_{i,j}(+) e^{-\beta H}} = \exp(\beta h_{i,j}(\sigma_i - 1)) . \quad (7)$$

Therefore [since the factors $\prod_{i,j} \exp(-h_{i,j})$ cancel] we have

$$\beta h_{n+1} = -\frac{1}{2} \ln \frac{\sum_{i=1}^{p-1} \sum_{\sigma_i} \exp \left[-\beta J_{n+1} \prod_{i=1}^{p-1} \sigma_i + \beta \sum_{i=1}^{p-1} (H + h_i) \sigma_i \right]}{\sum_{i=1}^{p-1} \sum_{\sigma_i} \exp \left[+\beta J_{n+1} \prod_{i=1}^{p-1} \sigma_i + \beta \sum_{i=1}^{p-1} (H + h_i) \sigma_i \right]} , \quad (8)$$

where $h_i = \sum_{j=1}^{q-1} h_{i,j}$. The meaning of h_i and $h_{i,j}$ is rather obvious: $h_{i,j}$ is the contribution of the j th n th-generation branch attached to the spin σ_i of the base polygon, altogether resulting in a random field contribution h_i exerted from the higher levels of the branch onto the spin σ_i . The effective random field h_{n+1} of the base spin can therefore be calculated recursively if the fields stemming from the $(p-1)(q-1)$ n th-generation branches are known. This means that starting with a given realization of random fields at the boundary [or equivalently with certain probability distributions for the boundary spins themselves, leading via $x = e^{2\beta H} P(-)/P(+)$ to a distribution of the effective fields], one has an iterative process to calculate the effective field h at the base spin, whose magnetization is then given by

$$\langle \sigma_{\text{base}} \rangle = \tanh[\beta(H + h)] . \quad (9)$$

If one completes the tree as described above by connecting together q n th-generation branches, the magnetization of the central spin is simply given by

$$\langle \sigma_0 \rangle = \tanh \left[\beta \left[H + \sum_{i=1}^q h_i \right] \right] . \quad (10)$$

Suppose now that the probability distribution of the fields at the boundary is given by the requirement that the iterative process preserves that contribution; one gets the following coupled integral equations for this probability distribution¹¹:

$$P(h) = \int \prod_{i=1}^{p-1} [g(h_i) dh_i] \times \langle \delta[h - \beta^{-1} \mathcal{F}_{\beta J, \beta H}(h_1, \dots, h_{p-1})] \rangle_J , \quad (11)$$

$$g(h_i) = \int \prod_{j=1}^{q-1} [P(h_{i,j}) dh_{i,j}] \delta \left[h_i - \sum_{j=1}^{q-1} h_{i,j} \right] , \quad (12)$$

where $\mathcal{F}_{\beta J, \beta H}(h_1, \dots, h_{p-1})$ is given by the right-hand side (rhs) of (8) and the angular brackets mean average over the distribution $\rho(J)$ of couplings. The Fourier transforms of g and P are simply connected via (12): $\bar{g}(k) = [\bar{P}(k)]^{q-1}$, but unfortunately even integrating out the δ function in (11) leaves us with $p-2$ integrals on the rhs of (11). Therefore the work that was done in the case of the Bethe lattice¹¹ ($p=2$) cannot be generalized in a straightforward manner.

III. SMALL COORDINATION NUMBERS

Instead of trying to solve Eqs. (11) and (12) numerically, we take a look at the self-consistency equations for the moments of the distribution of h that occur if we expand the rhs of (8) in powers of the fields. For simplicity we consider the case $p=3$, $q=3$ and vanishing external field ($H=0$). In this case (8) becomes

$$h = \frac{1}{2} \ln \frac{e^{\beta J} \cosh(h_1 + h_2) + e^{-\beta J} \cosh(h_1 - h_2)}{e^{-\beta J} \cosh(h_1 + h_2) + e^{\beta J} \cosh(h_1 - h_2)} , \quad (13)$$

where $h_1 = h_{11} + h_{22}$ and $h_2 = h_{21} + h_{22}$. Expanding in the fields one gets

$$h = \tanh(\beta J) h_1 h_2 + \frac{1}{3} \tanh(\beta J) (h_1^3 h_2 + h_1 h_2^3) + \mathcal{O}(h_a^5 h_b + h_a^3 h_b^3) ,$$

and therefore for the moments (note that h_1 and h_2 are uncorrelated),

$$\begin{aligned}\langle h^2 \rangle &= 10t_1 \langle h^2 \rangle^2 + \frac{8}{3} \langle h^4 \rangle, \\ \langle h^4 \rangle &= 4t_2 \langle h^4 \rangle^2 + (24t_2 + 30t_1) \langle h^4 \rangle \langle h^2 \rangle^2 \\ &\quad + t_2 \langle h^2 \rangle^4 + 2t_1 \langle h^6 \rangle \langle h^2 \rangle,\end{aligned}\quad (14)$$

where $t_n = \langle \tanh^{2n}(\beta J) \rangle_J$. Neglecting the term with $\langle h^6 \rangle$, one gets only $\langle h^2 \rangle = 0$ as a solution, in contrast to the Bethe lattice, where one has a critical value of t_1 , below which a stable solution $\langle h^2 \rangle \neq 0$ occurs.¹² Thus, assuming a second-order transition, like in the SK model, one concludes that there is no spin-glass phase. But mean-field theory predicts a first-order transition, where the EA order parameter jumps discontinuously at T_c , which means that the second moment of the distribution of internal fields also behaves discontinuously. The above expansion is one for small fields, and one cannot get information about a possible first-order transition, since there is no other small quantity (as, for instance, in a $p = 2 + \epsilon$ expansion⁶).

For zero temperature (13) simplifies to (note that $J > 0$)

$$\begin{aligned}h &= \text{sgn}(|h_1 + h_2 + 2H| - |h_1 - h_2|) \\ &\quad \times \min\{J, \frac{1}{2}||h_1 + h_2 + 2H| - |h_1 - h_2||\},\end{aligned}\quad (15)$$

where H is again the external field for later reference, but at the moment H is still zero. We insert the rhs of (15) for the function $\beta^{-1} \mathcal{F}_{\beta J, \beta H}$ in (11), but since it is hard to solve (11) analytically even in this case for a given $\rho(J)$, we try to find a distribution $\rho(J)$, when the distribution of fields is given.¹⁴ For $\alpha > 0$, one gets

$$\int_{\alpha}^{\infty} dJ \rho(J) = \frac{\int_{\alpha}^{\infty} dh P(h)}{\int_{\alpha}^{\infty} dh_1 \int_{\alpha}^{\infty} dh_2 g(h_2)g(h_1)}.\quad (16)$$

We tried to obtain a non-negative distribution $\rho(J)$ for several functions $P(h)$, but this was not possible. For example, $P(h) = e^{-2|h|}$ leads to

$$\int_{\alpha}^{\infty} dJ \rho(J) = e^{2\alpha} (1 + \alpha)^{-2},\quad (17)$$

and hence to a range of J , where $\rho(J)$ becomes negative. In what follows we will see that this is a typical result, since $P(h) = \delta(h)$ is the only solution of (15) for different classes of distributions $\rho(J)$.

Starting with a randomly chose set of fields at the boundary (e.g., distributed uniformly or according to a Gaussian) we iterated (15) numerically with up to eight levels. This was done up to 10 000 times and the second moment $\langle h^2 \rangle$ of the resulting distribution of fields at the base spin was calculated. The results for $H = 0$ are depicted in Fig. 2 for different distributions $\rho(J)$ of the interactions. We see that $\langle h^2 \rangle$ decays to zero with increasing number of levels, the decay rate slightly dependent on the specific distribution $\rho(J)$. For comparison we did the same for the Cayley tree with $q = 3$, in which case the iteration equation for zero temperature becomes

$$h = \text{sgn}(h_1 + h_2) \min\{J, |h_1 + h_2|\},\quad (18)$$

and obtained the expected result that $\langle h^2 \rangle$ decays to a nonvanishing limit for increasing number of levels, which

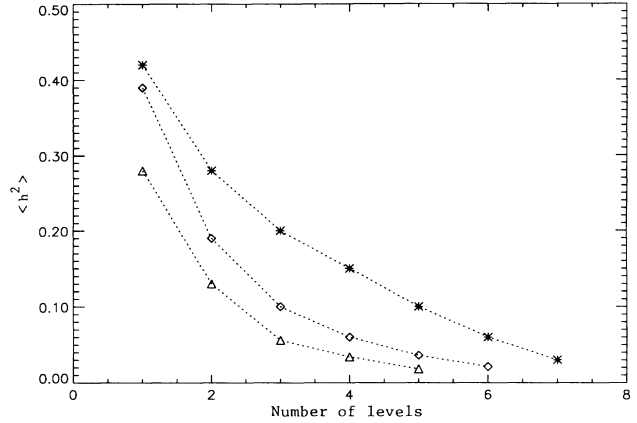


FIG. 2. Second moment $\langle h^2 \rangle$ of the distribution of fields at the base spin in dependence of the number of levels for zero temperature and field, $p = 3$, $q = 3$. The distribution of interactions $\rho(J)$ (before the gauge transformation mentioned in the text) are: stars for a ± 1 distribution, diamonds for a uniform distribution between -2 and $+2$, triangles for a Gaussian distribution with variance 1.

indicates the spin-glass behavior in the case of the Bethe lattice. Thus we get the surprising result that on the Husimi tree (with $p = 3$, $q = 3$) there is no spin glass, although frustration is present. We performed the above numerical procedure also for higher values of p (≤ 6) and q (≤ 8), for zero temperature as well as for finite temperature, using (8). No qualitative differences occur, but it has to be noted that the number of levels that can be explored numerically decreases rapidly with increasing p and q (mainly because of storage problems), which makes the results less and less reliable.

Furthermore we investigated the situation in a nonvanishing field. Using (15) (i.e., zero temperature and $p = 3$, $q = 3$) we obtained results, part of which are depicted in Fig. 3. Now we get a nonvanishing value of $\langle h^2 \rangle$, but to check whether this is due to a possible spin-glass phase or due to an ordering along the external field, we iterated (15) simultaneously for the same realization of interaction but starting with different boundary conditions,¹² resulting in two fields h_a and h_b at the base spins. The case $\langle h^2 \rangle \neq 0$ and $\langle h_a h_b \rangle = 0$ would indicate a spin-glass behavior, but a look at Fig. 3 shows us that after an intermittency regime both correlations increase to the same value $\langle h^2 \rangle = \langle h_a h_b \rangle = 1$ (this is obvious for $H \geq 0.5$; for lower values of H we have to extrapolate to higher numbers of levels, which we did not explore numerically). Indeed, $P(h) = \delta(h - 1)$ is a self-consistent solution of (15) for a ± 1 distribution of interaction; that means $J = 1$ in (15). We obtained similar results for other distributions as well as for higher values of p and q . In other words: an external field leads to a fully magnetized state deeper inside a large enough system. This ferromagnetic phase persists for temperatures below a critical value dependent on the external field. We leave it as an open question, whether the occurring phase transition to a paramagnetic phase is then still of first order for a certain range of external fields (as in the isotropic case¹⁵) or whether it is of second order.

IV. INFINITE COORDINATION NUMBER

To obtain a well-defined $q \rightarrow \infty$ limit, one has first (as in the infinite range models) to rescale the interaction strengths via $J \rightarrow J/\sqrt{q-1}$. To see how a simplification occurs, one has to recall the definition of the fields h_i on the rhs of (13):

$$h_i = \sum_{j=1}^{q-1} h_{i,j} . \tag{19}$$

In the limit of infinite coordination number $q \rightarrow \infty$, ac-

$$\bar{\beta} \langle h^2 \rangle = \int \frac{dx}{\sqrt{2\pi}} \int \frac{dy}{\sqrt{2\pi}} \exp \left[-\frac{x^2+y^2}{2} \right] \int dJ \rho(J) f^2(\bar{\beta}, x\beta\sqrt{\langle h^2 \rangle}, y\beta\sqrt{\langle h^2 \rangle}) , \tag{20}$$

where $f(\bar{\beta}J, h_1, h_2)$ is given by the rhs of (13), replacing βJ by $\bar{\beta}J$ and $\bar{\beta} = \beta/\sqrt{q-1}$. The paramagnetic solution ($\langle h^2 \rangle = 0$) is always possible and if a transition to a nonvanishing expectation value of the second moment of the field distribution occurs, then it must be a discontinuous one, since expanding the rhs of (20) yields

$$\begin{aligned} \langle h^2 \rangle &= \langle \tanh^2(\bar{\beta}J) \rangle_J (q-1)\beta^4 \langle h^2 \rangle^2 + O(\langle h^2 \rangle^3) , \\ &\rightarrow \beta^6 \langle J^2 \rangle \langle h^2 \rangle^2 + O(\langle h^2 \rangle^3) \end{aligned}$$

as $q \rightarrow \infty$, with $\beta \ll \sqrt{q-1}$. (21)

For simplicity we focused on a $\pm J$ distribution $\rho(J)$ and solved (20) numerically. In fact one finds a nonvanishing solution for $\langle h^2 \rangle$ at $T_c \approx 0.39J$, where $\langle h^2 \rangle \approx 0.15$. The critical temperature one finds in the mean-field theory (or infinite-range version) of the $p=3$ multisite interaction spin glass within the replica-symmetric theory⁶ (and the spherical approximation), is $T_c \approx 0.47J$, where the

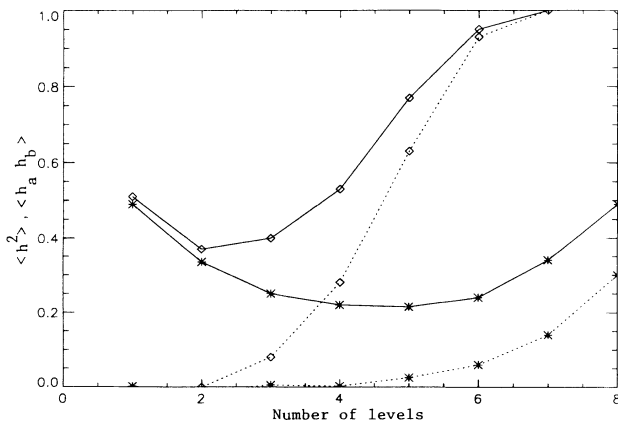


FIG. 3. Second moment $\langle h^2 \rangle$ (solid lines) and the correlation $\langle h_a h_b \rangle$ (dotted lines) for zero temperature; $H=0.25$ (stars) and $H=0.5$ (diamonds), and $p=3$, $q=3$ in case of ± 1 distribution of interactions before the gauge transformation.

ording to the central limit theorem the random field h_i becomes a Gaussian variable with mean square $(q-1)\langle h^2 \rangle$, where $\langle h^2 \rangle$ is the second moment of the probability distribution of the random fields. This distribution need not to be a Gaussian, nevertheless, (8) provides us with a self-consistency equation for its second moment by replacing the random fields h_i ($i=1,2,\dots,p-1$) by Gaussian random variables, taking the square on both sides of (8) and integrating the rhs over the $p-1$ Gaussian random variables. This yields the expectation value $\langle h^2 \rangle$. We demonstrate this idea for the case $p=3$. One gets then

Edwards-Anderson order parameter jumps discontinuously to a nonvanishing value.

In mean-field theory the replica-symmetric solution is unstable, the phase transition temperature is somewhat different due to replica-symmetry breaking effects and, furthermore, the equilibrium phase transition is preceded by a dynamical transition, where spin correlations do not decay on finite time scales. The fact that we chose a $\pm J$ distribution instead of a Gaussian distribution is without consequence for the comparison with the mean-field results, since these only depend on the first two cumulants of the distribution, because of the scaling of the interactions with the system size. With decreasing temperature $\langle h^2 \rangle$ increases monotonically (nearly linear) to $\langle h^2 \rangle = 1$ for zero temperature. Whether the system is in the spin-glass phase indicated by $\langle h^2 \rangle \neq 0$ below T_c depends on the variance of the fields at the boundary, which has to be large enough. The absence of replica-symmetry breaking effects in our treatment is connected to the choice of uncorrelated boundary conditions.¹⁶ It would be interesting to check how one can recover the mean-field solution with replica-symmetry breaking in the case of Husimi trees.

V. DISCUSSION

The most unusual feature of the results we obtained is the fact that the existence of a spin-glass transition is dependent on the coordination number of the Husimi tree. This dependence is completely absent in the Bethe approximation to the Edwards-Anderson model, which predicts a spin-glass transition for any coordination number of the tree. Presumably the degree of frustration in the Husimi tree is not high enough to lead to a spin-glass phase for small coordination number and increases for larger values.

Since this number is proportional to the number of nearest neighbors ($z=pq$), it can be interpreted as the dependence of the existence of the phase transition on the dimensionality of spin systems with random p -spin in-

teractions, supposing one accepts the Husimi tree as a reliable approximation to these models. Indeed one can show that in several two-dimensional systems a spin-glass transition cannot occur, and there are indications from numerical simulations that, in the (three-dimensional) fcc lattice with random four-spin interactions among spins on the elementary tetrahedra, an equilibrium phase transition is either absent or is preceded by a dynamical transition, and therefore can hardly be observed in simulations within reasonable computer time.⁹ Up to now it is an open question whether a finite coordination number q_c exists, above which the mean-field scenario takes place, or whether q_c is infinite. It might be possible to settle this question by a $1/q$ expansion. Furthermore, as already mentioned, the behavior of the system is extremely sensitive to the choice of the boundary conditions. Throughout this paper we used uncorrelated (but fixed) boundary conditions. Using correlated (or closed) boundary conditions¹⁶ could have some effect on q_c and perhaps even lead to a spin-glass transition for small q .

This problem is of course related to the unknown upper critical dimension of random multisite interaction spin-glass models. It is not obvious to formulate a field theory for these models in finite dimensions, because there is much arbitrariness (and presumably some sensitivity of the theoretical predictions) to the definition of the p -spin interactions among nearest neighbors. Furthermore—supposing that one has such a formulation for, e.g., $p=3$ —it is questionable whether the usual expansion in the order-parameter fields in the vicinity of the critical point is meaningful, since one would expect a discontinuous phase transition. Hence one needs a theory in finite dimensions, where one is able to perform an analytical continuation to noninteger values of p , so that a further small parameter, namely $\epsilon=p-2$, becomes available.

Similar difficulties are encountered in Potts glasses. The mean-field results indicate that the p -state Potts glass (for $p>4$) and the p -spin interaction spin glasses (for $p>2$) are in the same universality class.⁶ But results for Potts glasses on the Bethe lattice¹⁷ differ from those we

obtained for p -spin interaction models on Husimi trees, since the former shows a spin-glass transition analogous to the mean-field scenario. Therefore it is an important issue to clarify whether both models are in different universality classes for finite dimensions, or whether a range of dimensions exists, where both are still within the same.

We would like to draw some attention to another point. In mean-field theory the limit $p\rightarrow\infty$ drastically simplifies the model—it becomes the random energy model, which can be solved exactly.⁴ Such a simplification does not occur on Husimi trees, mainly because spin-spin correlation functions do not enter the recursion formulas for the field by which we investigate the system. Therefore it might be desirable to obtain a different formulation of the problem, perhaps similar to the one used in the case of the Potts glass on the Bethe lattice.¹⁷

Concluding, we mention that in this paper we addressed only the static behavior of the random p -spin interaction models on Husimi trees. An important point, especially with respect to the dynamical models of the structural glass transition (see the Introduction), is also the investigation of the dynamics of these models. One could, for instance, think of calculating quantities such as the remanent magnetization with methods successfully applied to the spin glass on the Bethe lattice.¹⁸ Furthermore, one cannot exclude that these models show a dynamical phase transition, where relaxation times diverge, although no equilibrium phase transition occurs. We will address several of the important issues mentioned in this discussion in future work.

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