The number of solutions of the Thouless-Anderson-Palmer equations for p-spin-interaction spin glasses

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The number $\langle N_s \rangle$ of solutions of the equations of Thouless, Anderson, and Palmer for p-spin-interaction spin glass models is calculated. Below a critical temperature T_c this number becomes exponentially large, as it is in the Sherrington-Kirkpatrick model ($p = 2$). But in contrast to this, for any $p > 2$ the factor $\alpha(T) = N^{-1} \ln\langle N_s \rangle$ jumps discontinuously at $T_c(p)$, which is consistent with the discontinuity occurring within the mean-field theory for these models. For zero temperature the results obtained by Gross and Mézard are reproduced, and for $p \to \infty$ one gets the result for the random energy model.

I. INTRODUCTION

The free energy surface of a spin glass[1] has an extremely complicated structure, which manifests itself in the presence of many valleys with high barriers between them. Below a critical temperature T_c the number of valleys becomes exponentially large and the barriers diverge with system size (at least in the mean–field limit). The local mean-field equations for the site magnetizations, originally written down by Thouless, Anderson, and Palmer (TAP) (Ref. 2) in an effort to solve the Sherrington-Kirkpatrick (SK) model[3] without using replicas, give rise to an exponentially large number N_s of solutions below T_c. In the case of the SK model it was shown[4] that $N_s \propto \exp[\alpha(T)\, N]$, where $\alpha(T)$ increases very slowly $[\alpha(T) \propto (T - T_c)^6]$ below T_c. To recover the Parisi solution[5] of the SK model one has to attribute Boltzman weights to each of these solutions before taking the average over the bond distribution,[6] since only those with the lowest free energies contribute significantly to the thermodynamics.

The spin-glass transition occurring in the mean-field theory of p-spin-interaction spin glasses[7–10] is somewhat different for $p > 2$ (for a discussion of finite-dimensional realizations of these models see Refs. 11 and 12). The equilibrium phase transition at a temperature T_K manifests itself by a discontinuous jump in the Edwards-Anderson (EA) order parameter (although the transition itself is of second order). But already at a higher temperature $T_g > T_K$ a discontinuous dynamical freezing transition takes place, where spin autocorrelations do not decay anymore on finite time scales.[13] Within the TAP approach this means that at T_g an exponentially large number of solutions has to appear, which are uncorrelated and have a higher free energy than the paramagnetic state — similar to what happens in p-state Potts glasses with $p > 4$.[14]

In this paper we generalize the above-mentioned calculation of Bray and Moore[4] to the case of p-spin-interaction spin glasses. We perform a white average,[15] which means that we attribute the same weight to all solutions, so one cannot discuss the thermodynamics of these models on the basis of our calculation. Nevertheless it will be interesting to observe that the temperature dependence of $N_s(T)$ is essentially different for $p > 2$ from that of the SK model — in a way that is consistent with a discontinuous transition. The organization of the paper is as follows: In Sec. II we formulate the problem and derive the self–consistency equations, which are solved in Sec. III, where also the results are presented. Section IV contains their discussion and two lengthy calculations are deferred to the appendixes.

II. NUMBER OF TAP SOLUTIONS

The Hamiltonian for p-spin-interaction spin glasses within the mean-field approximation reads

$$\mathcal{H} = -\sum_{i_1<\cdots<i_p} J_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p} - H\sum_i \sigma_i \,, \qquad (1)$$

$\sigma_i = \pm 1$, $i = 1, \ldots, N$ and H is an external field. The p-spin couplings are quenched random variable distributed according to a Gaussian

$$\mathbf{P}(J_{i_1\cdots i_p}) = \frac{1}{\sqrt{\pi}\tilde{J}}\exp\left(-\frac{J^2_{i_1\cdots i_p}}{\tilde{J}^2}\right), \quad \tilde{J}^2 = \frac{J^2 p!}{N^{p-1}} \,. \qquad (2)$$

Each spin is subject to a local field of strength

$$h_i = \frac{1}{(p-1)!}\sum_{j_2,\ldots,j_p} J_{ij_2\cdots j_p}\sigma_{j_2}\cdots\sigma_{j_p} + H \,. \qquad (3)$$

Hence — in analogy to Ref. 2 — the TAP equations are

$$m_i = \tanh\beta\left(\frac{1}{(p-1)!}\sum_{j_2,\ldots,j_p} J_{ij_2\cdots j_p} m'_{j_2}\cdots m'_{j_p} + H\right) , \qquad (4)$$

with $m'_j = m_j - \chi_{jj}\Delta'_j$, which is the magnetization of site j minus the Onsager–correction term (or cavity field). χ_{jj} is the susceptibility of spin j:

$$\chi_{jj} = \beta(1 - m_j^2) \tag{5}$$

and $\Delta h'_j$ is the field induced at site j by the magnetization of site i

$$\Delta h'_j = m_i \cdot \frac{1}{(p-2)!} \sum_{k_3,\ldots,k_p} J_{ijk_3\cdots k_p} m_{k_3} \cdots m_{k_p} \,. \tag{6}$$

Inserting this expression into Eq. (4) yields a rather complicated structure for the TAP equations with terms up to order p in the couplings $J_{i_1,\cdots,i_p}$. In Appendix A we show that neglecting terms of order $O(N^{-1})$ one gets

$$\tanh^{-1}(m_i) = \frac{\beta}{(p-1)!} \sum_{j_2,\ldots,j_p} J_{ij_2\cdots j_p} m_{j_2} \cdots m_{j_p} + \beta H - m_i \frac{(\beta J)^2}{2} p(p-1)(1-q)q^{p-2} \,, \tag{7}$$

where we defined the self-overlap of a paricular TAP solution

$$q = \frac{1}{N} \sum_i m_i^2 \,. \tag{8}$$

One can obtain Eq. (7) by differentiation of the free-energy functional f with respect to m_i:

$$\beta f = \frac{1}{2N} \sum_i \left\{ (1+m_i) \ln\left(\frac{1+m_i}{2}\right) + (1-m_i) \ln\left(\frac{1-m_i}{2}\right) \right\} - \frac{\beta}{N} \sum_{i_1<\cdots<i_p} J_{i_1\cdots i_p} m_{i_1} \cdots m_{i_p} - \frac{(\beta J)^2}{4} [(p-1)q^p - pq^{p-1} + 1] \,. \tag{9}$$

The number of solutions N_s of the TAP equations Eq. (7) is given by

$$N_s = N \int_0^1 dq \int_{-1}^{+1} \prod_i dm_i \, \delta\left(Nq - \sum_i m_i^2\right) \prod_i \delta(G_i) \, |\det \underline{A}| \,, \tag{10}$$

where

$$G_i = g(m_i) - \frac{\beta}{(p-1)!} \sum_{j_2,\ldots,j_p} J_{ij_2\cdots j_p} m_{j_2} \cdots m_{j_p} \,, \tag{11}$$

$$g(m_i) = \tanh^{-1}(m_i) + m_i \frac{(\beta J)^2}{2} p(p-1)(1-q)q^{p-2} - \beta H \,, \tag{12}$$

$$A_{ij} = \frac{\partial G_i}{\partial m_j} = a_i \delta_{ij} - \frac{\beta}{(p-2)!} \sum_{k_3,\ldots,k_p} J_{ijk_3\cdots k_p} m_{k_3} \cdots m_{k_p} \,, \tag{13}$$

$$a_i = \frac{1}{1-m_i^2} + \frac{(\beta J)^2}{2} p(p-1)(1-q)q^{p-2} \,. \tag{14}$$

Following Ref. 4 we calculate $\langle N_s \rangle$ — which means the average of N_S over the distribution of the couplings (2) — and discuss the implication of the fact that one should rather introduce replicas and perform the average $\ln\langle N_s \rangle$ in Sec. IV:

$$\langle N_s \rangle = N \int \frac{dq\, d\hat{q}}{2\pi} \int \prod_i \frac{dm_i\, d\hat{m}_i}{2\pi} \exp\left\{ \sum_i \hat{m}_i g(m_i) + \hat{q}\left(Nq - \sum_i m_i^2\right) \right\} \tag{15}$$

$$\times \left\langle \exp\left\{ \frac{\beta}{(p-1)!} \sum_{i_1,\ldots,i_p} J_{i_1\ldots j_p} \hat{m}_{i_1} m_{i_2} \cdots m_{i_p} \right\} \det(\underline{A}) \right\rangle \,. \tag{16}$$

The last factor can be written as

$$\langle \cdots \rangle = \prod_{i_1<\cdots<i_p} \exp \frac{(\beta \tilde{J})^2}{4(p-2)!} \left(\sum_\pi \hat{m}_{\pi(i_1)} m_{\pi(i_2)} \cdots m_{\pi(i_p)} \right)^2 \langle \det \underline{A}' \rangle \,, \tag{17}$$

where $\sum_\pi$ is a sum over all permutations of p different integers $i_1, \ldots, i_p$ and

$$A'_{ij} = a_i - \frac{\beta}{(p-2)!} \sum_{k_3,\ldots,k_p} J_{ijk_3\cdots k_p} m_{k_3} \cdots m_{k_p} + \frac{(\beta J)^2 p}{(p-2)! N^{p-1}} \sum_{k_3,\ldots,k_p} \sum_\pi \hat{m}_{\pi(i)} m_{\pi(j)} m_{\pi(k_3)} \cdots m_{\pi(k_p)} \, m_{k_3} \cdots m_{k_p} \,. \tag{18}$$

The first sum yields a Gaussian random variable of order $O(1/\sqrt{N})$, whereas the second sum produces three kinds of terms: $N^{-1}(\hat{m}_i m_j + m_i \hat{m}_j)q^{p-2}$, $N^{-1}(\hat{m}_i m_j + m_i \hat{m}_j)q^{p-4}M^2$, and $N^{-1}m_i m_j q^{p-3}G$, where $M = N^{-1}\sum_i m_i$ and $G = N^{-1}\sum_i m_i \hat{m}_i$. They are all proportional to N^{-1} and can therefore be neglected, which implies $\underline{A}' = \underline{A}$. The squared sum in Eq. (17) is treated as follows:

$$\sum_{i_1,\ldots,i_p} \sum_{\pi} \hat{m}_{\pi(i_1)} m_{\pi(i_2)} \cdots m_{\pi(i_p)} \hat{m}_{i_1} m_{i_2} \cdots m_{i_p} = (p-1)! \left\{ N^{p-1} q^{p-1} \sum_i \hat{m}_i^2 + (p-1) N^{p-2} q^{p-2} \left(\sum_i \hat{m}_i m_i \right)^2 \right\} . \tag{19}$$

Using a Hubbard Stratonoviç transformation to linearize the squared term we get

$$\langle N_s \rangle = c \int dy \int dq\, d\hat{q} \int \prod_i \frac{dm_i d\hat{m}_i}{2\pi} \exp\left\{ \sum_i \hat{m}_i g(m_i) + \hat{q}\left(Nq - \sum_i m_i^2 \right) \right\}$$
$$\times \exp\left\{ \frac{(\beta J)^2}{4} p q^{p-1} \sum_i \hat{m}_i^2 + y \sum_i m_i \hat{m}_i - \frac{N y^2}{(\beta J)^2 p (p-1) q^{p-2}} \right\} \langle \det \underline{A} \rangle . \tag{20}$$

In Appendix B we derive an expression for $\langle \det \underline{A} \rangle$, the above integration will be done by steepest descent (which becomes exact in the limit $N \to \infty$) and we are left with

$$\langle N_s \rangle = \mathcal{S}_{y,z,q,\hat{q}} \exp N \left(\hat{q} q - \frac{y^2}{\lambda} + \frac{z^2}{\lambda} \right)$$
$$\times \left\{ \int_{-1}^{+1} dm \int_{-\infty}^{\infty} \frac{d\hat{m}}{2\pi} \exp\left[i\hat{m} g(m) - \hat{q} m^2 + \frac{\mu}{2} q^{p-1} (i\hat{m})^2 + y m\, i\hat{m} \right] (a - z) \right\}^N , \tag{21}$$

where $\mathcal{S}$ means "saddle point" and $\mu = (\beta J)^2 p/2$, $\lambda = 2\mu(p-1)q^{p-2}$ and $a = (1-m^2)^{-1} + (1-q)\lambda/2$. For simplicity we set the external field H to zero from now on. Introducing the variables used in Ref. 4

$$B = z - \lambda(1-q)/2 ,$$
$$\Delta = y + \lambda(1-q)/2, \tag{22}$$

and performing the integration over $\hat{m}$ (which is Gaussian) we get

$$\langle N_s \rangle = \mathcal{S}_{B,\Delta,q,\hat{q}} \exp N \left[-q\hat{q} - (B+\Delta)(1-q) + \frac{B^2 - \Delta^2}{2\mu(p-1)q^{p-2}} + \ln I \right] , \tag{23}$$

$$I = \int_{-1}^{+1} \frac{dm}{\sqrt{2\pi\mu q^{p-1}}} \left(\frac{1}{1-m^2} + B \right) \exp\left\{ -\frac{[\tanh^{-1}(m) - \Delta m]^2}{2\mu q^{p-1}} + \hat{q} m^2 \right\} , \tag{24}$$

which reduces in the special case $p = 2$ to Eqs. (15)–(17) of Ref. 4. The saddle-point equations again admit the solution $B = 0$, which we adopt here. We define $\sigma = \sqrt{\mu q^{p-1}}$, $\tilde{\Delta} = \Delta/\sigma$ and use the variable substitution $x = \tanh^{-1}(m)/\sigma$ to proceed further. Then the saddle-point equations for q, $\tilde{\Delta}$ and $\hat{q}$ read

$$q = \frac{1}{I'} \int_0^\infty dx\, [\tanh(\sigma x)]^2 \exp[L(\sigma, \tilde{\Delta}, \hat{q})] ,$$
$$\tilde{\Delta} = \frac{p-1}{pq} \frac{1}{I'} \int_0^\infty dx\, x \tanh(\sigma x) \exp[L(\sigma, \tilde{\Delta}, \hat{q})] - p^{-1}\mu(1-q)q^{p-2}(p-1) , \tag{25}$$
$$\hat{q} = \left\{ \frac{1}{I'} \int_0^\infty dx\, [x - \tilde{\Delta}\tanh(\sigma x)]^2 \exp[L(\sigma, \tilde{\Delta}, \hat{q})] - 1 \right\} \frac{p-1}{2q} + \sigma\tilde{\Delta} + \frac{p-2}{p-1}\frac{\tilde{\Delta}^2}{2} ,$$

where

$$L(\sigma, \tilde{\Delta}, \hat{q}) = -\frac{1}{2}[x - \tilde{\Delta}\tanh(\sigma x)]^2 + \hat{q}\, [\tanh(\sigma x)]^2 ,$$
$$I' = \int_0^\infty dx\, \exp[L(\sigma, \tilde{\Delta}, \hat{q})] . \tag{26}$$

Since $\tilde{\Delta}$ is well behaved for $T \to 0$ the quantity Δ diverges, which is the main reason for this substitution. For the numerical solution of the saddle-point equations it is also advisable to use $(\hat{q} - \Delta)$ instead of $\hat{q}$, since the latter also diverges for $T \to 0$.

III. RESULTS

First we look at the the zero-temperature limit. For $\beta \to \infty$ the quantity μ will diverge also and therefore $\sigma \to \infty$, too. Hence we may replace $\tanh(\sigma x)$ by 1 within the integrals, since they only extend over $x > 0$. This yields (as expected) $q = 1$ and furthermore, after a shift

$$y = x - \tilde{\Delta}$$

$$I' = e^{\hat{q}} \int_{-\tilde{\Delta}}^{\infty} dy\, e^{-y^2/2} \tag{27}$$

and hence

$$\tilde{\Delta} = (p-1) \cdot \frac{e^{-\tilde{\Delta}^2}}{\int_{-\tilde{\Delta}}^{\infty} dy\, e^{-y^2/2}} . \tag{28}$$

With the above-defined quantities (and $I = I'\sqrt{2/\pi}$) we get for $T = 0$

$$\langle N_s \rangle = \exp N \left\{ \ln 2 - \frac{\tilde{\Delta}^2}{2(p-1)} + \ln \int_{-\tilde{\Delta}}^{\infty} dy\, e^{-y^2/2} \right\} , \tag{29}$$

where $\tilde{\Delta}$ has to be determined via Eq. (28). This is in complete agreement with what was found by Gross and Mézard (Ref. 8). For $T = 0$ the number of TAP solutions increases monotonically from $\ln N_s = 0.1992N$ for $p = 2$ (see also Ref. 16) to $\langle N_s \rangle \approx 2^N$ for $p \to \infty$ (the random energy model).

The exploration of the behavior of N_s for nonvanishing temperatures has to be done numerically. The success of numerical methods for solving nonlinear equations like Eq. (25) rely heavily on the quality of an initial guess. Therefore we started at $T = 1 - t$ with $t \ll 1$ (we set $J = 1$ from now on) for $p = 2$, where one has an analytic expression for q, $\tilde{\Delta}$ and $\hat{q}$ in form of a power series in t.[4] Then we decreased the temperature by small steps δT, always using $q(T)$, $\Delta(T)$, and $\hat{q}(T)$ as an initial guess for $q(T - \delta T)$, $\Delta(T - \delta T)$, and $\hat{q}(T - \delta T)$. This was done down to very small temperatures and then we fixed the temperature and increased p by small steps δp up to $p = 3$ using the same procedure. Finally we fixed $p = 3$ and increased the temperature again, up to a value, where the solution disappeared discontinuously. In the same way we solved Eq. (25) for higher values of p. Numerically it is much more difficult to start with $p = 2$ and a temperature near $T = 1$ and then to increase p slowly, since $q = \tilde{\Delta} = \hat{q} = 0$ is always a stable solution for any temperature as long as $p > 2$. By the method we used we were sure to be on the same solution branch as the low-temperature solution for $p = 2$. We did not find other nontrivial solutions of Eq. (25) although they might exist.

The result for $\alpha(T) = N^{-1} \ln\langle N_s \rangle$ is shown in Fig. 1. At $T_c(p)$ the factor $\alpha(T)$ jumps discontinuously for any $p > 2$. Furthermore one observes that as long as there is an exponentially large number of TAP solutions it increases monotonically with p for a fixed temperature. With increasing p the critical temperature $T_c(p)$ first decays from $T_c = 1$ for the SK model ($p = 2$) to a value slightly below the critical temperature $T_c^{\rm RE}$ of the random energy model [$p \to \infty$: $T_c^{\rm RE} = (2\sqrt{\ln 2})^{-1} \approx 0.60$ (Ref. 7)]. For $p > 4$ it increases slowly to that value. Also $\langle N_s \rangle$ approaches for $p \to \infty$ the form predicted by the random energy model, which is a step function: $\alpha(T) = \ln 2 \cdot \theta(T_c^{\rm RE} - T)$. For $p \to \infty$ nearly every spin configuration of the system is a solution of the TAP equations.[8]

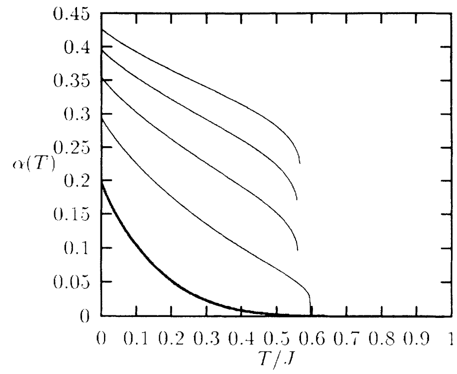

FIG. 1. The factor $\alpha(T) = N^{-1} \ln\langle N_s(T) \rangle$ in dependence of the temperature T for $p = 2$, 3, 4, 5, and 6 (from bottom to top). The curve for $p = 2$ is already known from Ref. 4.

In Fig. 2 we plotted the result of the mean-squared magnetization $q = \langle m^2 \rangle$ of the TAP solutions in dependence of the temperature. The curve for the SK model ($p = 2$) resembles that of the EA order parameter $q_{\rm EA}$ [or $q(x = 1)$ in Parisi's theory[5]] — but we point out that they are not the same, since we performed a white average over the solutions (see Sec. I). As long as there is an exponentially large number of TAP solutions q increases monotonically with p for a fixed temperature. Again we observe that in the limit $p \to \infty$ the result for the random energy model, where $q = 1$ below $T_c^{\rm RE}$ is approached. This means that in nearly every configuration of the system all spins are frozen below T_c.

From the last figure one recognizes that solutions of the TAP equations describe configurations of the system that are frozen to a large extent already for rather small values of p. This means that the spins are more or less fixed to values $+1$ or -1 and do not fluctuate significantly below $T_c(p)$. This feature is expected in the limit $p \to \infty$, but it is rather surprising that it is a good approximation for p as small as 5. In Ref. 8 it was argued that an expansion of the free energy and the order parameter function $q(x)$ around $p = \infty$ might be rapidly convergent, which means that already for rather small values of $p > 2$ the features of the random energy model might dominate the physics

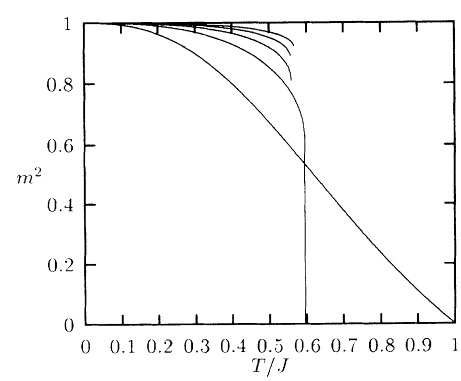

FIG. 2. The order paramterer $q = \langle m^2 \rangle$ (or mean-squared magnetizations of the TAP solutions) in dependence of the temperature T for $p = 2$, 3, 4, 5, and 6 (from bottom to top).

of p–spin-interaction spin glasses. Our results support this conjecture.

IV. DISCUSSION

After reducing the problem of calculating the number $N_s(T)$ of TAP solutions for p-spin-interaction spin glasses to a set of self–consistency equations, we showed that below a critical temperature this number becomes exponentially large: $N_s(T) \propto \exp[\alpha(T)\,N]$. For $p > 2$ the factor $\alpha(T)$ jumps discontinuously at the critical temperature T_c. This is consistent with existing mean–field theories of p–spin-interaction spin glasses, where the discontinuity is observed in the EA order parameter $q_{\rm EA}$. For increasing p the critical temperature approaches the (exactly known) value for the random energy model very rapidly and the shape of $\alpha(T)$ approaches the step function of the random energy model.

We also observe that the mean-squared magnetization $\langle m^2\rangle$ of the TAP solutions behaves similarly to $q_{\rm EA}$ and becomes equal to it in the limit $p \to \infty$. Nevertheless it should be noted that for finite p, $\langle m^2\rangle \neq q_{\rm EA}$, since we performed a white average, whereas $q_{\rm EA}$ can only be obtained by attributing a Boltzman weight to the TAP solutions. Furthermore, the discontinuity in $\alpha(T)$ is reminiscent of the jump in a quantity called "complexity" or "configurational entropy" in the context of metastable states in p-state Potts glasses with $p > 4$.[14]

The latter work, combined with the observation that the mean–field theory for p-spin-interaction spin glasses with $p > 2$ and that for p-state Potts glasses with $p > 4$ seem to be in the same universality class,[13] gives us reason to believe that performing the average over N_s — as we did — instead over $\ln(N_s)$ yields the right factor $\alpha(T)$ at least in a small temperature regime around T_c. In the Potts case it was shown[14] that in a temperature regime $T_K < T < T_g$ (T_g is the temperature, where the above-mentioned configurational entropy jumps discontinuously), the TAP solutions only have self–overlap. This means that calculating $\ln\langle N_s\rangle$ via replicas requires only diagonal order parameters (in replica space) for $T > T_K$.[14] Hence the behavior of $\alpha(T)$ around T_c — especially the discontinuity — will not be changed if one performs the correct average over $\ln\langle N_s\rangle$. At a lower temperature $T_K < T_c$, where the true equilibrium phase transition takes place, the picture might change slightly. It is desirable to undertake more detailed investigations on the above-mentioned points and work on the TAP-approach to the thermodynamics of p-spin-interaction spin glasses is in progress.[17]

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APPENDIX A

Here we derive Eq. (7) of the main text. Remembering Eqs. (4)–(6) we have

$$\frac{\beta}{(p-1)!}\sum_{j_2,\ldots,j_p} J_{ij_2\cdots j_p} m'_{j_2}\cdots m'_{j_p} = A - B + C\,, \tag{A1}$$

where we define

$$A = \frac{\beta}{(p-1)!}\sum_{j_2,\ldots,j_p} J_{ij_2\cdots j_p} m_{j_2}\cdots m_{j_p}\,, \tag{A2}$$

$$B = \frac{\beta^2}{(p-2)!^2}\sum_{j_2,\ldots,j_p} J_{ij_2,\ldots,j_p} m_i(1-m_{j_2}^2)\,m_{j_2}\cdots m_{j_p} \times \sum_{k_3,\ldots,k_p} J_{ij_2k_3\cdots k_p} m_{k_3}\cdots m_{k_p}\,. \tag{A3}$$

The remaining part C contains terms of higher order than second within the couplings. Note that we have already used the permutation symmetry of the couplings for the expression B. It can be written as

$$B = \frac{\beta^2 m_i}{(p-2)!^2}\sum_j (1-m_j^2) \times\left(\sum_{k_3,\ldots,k_p} J_{ijk_3\cdots k_p} m_{k_3}\cdots m_{k_p}\right)^2. \tag{A4}$$

The sum, which is squared is a sum over $M = N^{p-2}$ independent random variables $X_{j,1},\ldots,X_{j,M}$, whose variance is $\langle X^2\rangle \propto 1/N^{p-1}$. It can be splitted into two parts:

$$S_j = \left(\sum_{n=1}^{M} X_{j,n}\right)^2 = \underbrace{\sum_n X_{j,n}^2}_{\equiv:\,\overline{S}_j} + \underbrace{\sum_{n\neq m} X_{j,n}X_{j,m}}_{\equiv:\,\delta S_j}\,. \tag{A5}$$

The first summation $\overline{S_j}$ is a sum over positive random numbers of order $1/N^{p-1}$ and therefore yields a quantity of order $O(M/N^{p-1}) = O(1/N)$, whereas the second summation δS_j yields (by using the central limit theorem) a Gaussian random variable with mean zero and variance $M^2\langle X^2\rangle^2 = O(1/N^2)$. Hence B is

$$B = \frac{\beta^2 m_i}{(p-2)!^2}\sum_j (1-m_j^2)[\overline{S}_j + \delta S_j] = \overline{B} + \delta B\,. \tag{A6}$$

$\overline{B}$ is a term of order $O(N\overline{S}) = O(1)$, whereas δB is again a Gaussian variable with zero mean and variance $N\cdot\langle\delta S_j^2\rangle \propto 1/N$, which can be neglected with respect to $\overline{B}$. This leaves us with

$$B = \frac{\beta^2 m_i}{(p-2)!^2}\sum_j (1-m_j^2)(p-2)! \times \sum_{k_3,\ldots,k_p} J^2_{ijk_3\cdots k_p} m_{k_3}^2\cdots m_{k_p}^2\,, \tag{A7}$$

the factor $(p-2)!$ stems from the permutation symmetry of the couplings. Using

$$N^{-1}\sum_{k_p} J^2_{ijK_3\ldots k_p} m^2_{k_p} \approx \langle J^2_{ijk_3\cdots k_p}\rangle\langle m^2_{k_p}\rangle = \frac{J^2 p!}{2N^{p-1}} q\ , \tag{A8}$$

where we neglected again fluctuations around the average, which are of lower order in N, we finally end up with

$$B = \frac{(\beta J)^2}{2} p(p-1)(1-q)q^{p-2}\ . \tag{A9}$$

The part C on the rhs of Eq. (A1) is a sum over terms $(s \geq 2)$

$$T = \sum_{j_1,\ldots,j_s} (1-m^2_{j_1})\cdots(1-m^2_{j_s}) \left(\sum_{k_{s+1},\ldots,k_p} J_{ij_1\cdots j_s k_{s+1}\cdots k_p} m_{k_{s+1}}\cdots m_{k_p} \right)^{s+1}\ . \tag{A10}$$

The order of T is given by

$$O(T) = O\left(N^s \left[\sum_{n=1}^{M_s} x_n \right]^{s+1} \right)\ , \tag{A11}$$

where $M_s = N^{p-(s+1)}$ and X_n are again independent random variables with zero mean and variance $\langle X^2\rangle = 1/N^{p-1}$. Therefore the order of T cannot be greater than $O(N^{-s^2})$, which proves the correctness of Eq. (7) up to order $O(1/N)$.

APPENDIX B

Here we calculate $\langle \det \underline{A}\rangle$ by using the identity

$$\det \underline{A} = \lim_{n\to -2} \int \prod_{i=1}^{N}\prod_{\alpha=1}^{n} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\sum_{i\alpha} \xi_{i\alpha} A_{ij} \xi_{j\alpha} \right\}\ . \tag{B1}$$

From Eq. (13) in the main text we have then

$$\begin{aligned}\langle \det \underline{A}\rangle &= \left\langle \int \prod_{i,\alpha} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\sum_{i,\alpha} a_i \xi^2_{i\alpha} + \frac{\beta}{2(p-2)!} \sum_\alpha \sum_{i_1,\ldots,i_p} J_{i_1\cdots i_p} \xi_{i_1\alpha}\xi_{i_2\alpha} m_{i_3}\cdots m_{i_p} \right\} \right\rangle \\ &= \int \prod_{i,\alpha} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\sum_{i,\alpha} a_i \xi^2_{i\alpha} + \frac{(\beta\tilde{J})^2}{16(p-2)!^2} \sum_{i_1<\cdots<i_p} \left(\sum_{\pi,\alpha} \xi_{\pi(i_1)\alpha}\xi_{\pi(i_2)\alpha} m_{\pi(i_3)}\cdots m_{\pi(i_p)} \right)^2 \right\}\ ,\end{aligned} \tag{B2}$$

where $\sum_\pi$ means again the sum over all permutations of p different integers $i_1,\ldots,i_p$. After some algebra we get

$$\begin{aligned}\langle \det \underline{A}\rangle = \int \prod_{i,\alpha} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \exp\Bigg\{ &-\frac{1}{2}\sum_{i,\alpha} a_i \xi^2_{i\alpha} \\ &+ N\frac{(\beta J)^2}{16} p(p-1) \sum_{\alpha,\gamma} \Bigg[2q^{p-2} \left(\frac{1}{N}\sum_i \xi_{i\alpha}\xi_{i\gamma} \right)^2 \\ &+ 4(p-2)q^{p-3} \left(\frac{1}{N}\sum_i \xi_{i\alpha}\xi_{i\gamma} \right)\left(\frac{1}{N}\sum_i \xi_{i\alpha} m_i \right)\left(\frac{1}{N}\sum_i \xi_{i\gamma} m_i \right) \\ &+ (p-2)(p-3)q^{p-4} \left(\frac{1}{N}\sum_i \xi_{i\alpha} m_i \right)^2 \left(\frac{1}{N}\sum_i \xi_{i\gamma} m_i \right)^2 \Bigg] \Bigg\}\ .\end{aligned} \tag{B3}$$

To see that one can neglect the last two terms one introduces δ funtions for

$$\begin{aligned} z_{\alpha\gamma} &= \frac{1}{N}\sum_i \xi_{i\alpha}\xi_{i\gamma}\ , \\ x_\alpha &= \frac{1}{N}\sum_i \xi_{i\alpha} m_i, \end{aligned} \tag{B4}$$

with the help of conjugate variables $\hat{z}_{\alpha\gamma}$ and $\hat{x}_\alpha$, respectively (we do not pay attention to the fact that $z_{\alpha\gamma} = z_{\gamma\alpha}$

because it does not matter for this discussion).

$$\langle \det \underline{A} \rangle = N^2 \int \prod_{i,\alpha} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \int \prod_{\alpha,\gamma} \frac{dz_{\alpha\gamma} d\hat{z}_{\alpha\gamma}}{2\pi} \int \prod_{\alpha} \frac{dx_\alpha d\hat{x}_\alpha}{2\pi} \exp\left\{ -\frac{1}{2} \sum_{i,j} \sum_{\alpha,\gamma} \xi_{i\alpha} M_{ij,\alpha\gamma} \xi_{j\gamma} \right\} \times \exp N \left\{ \sum_{\alpha,\gamma} z_{\alpha\gamma} \hat{z}_{\alpha\gamma} + \sum_{\alpha} \hat{x}_\alpha x_\alpha + \frac{(\beta J)^2}{2} p(p-1) q^{p-2} \sum_{\alpha\gamma} z_{\alpha\gamma}^2 \right\}, \tag{B5}$$

where

$$M_{ij,\alpha\gamma} = ([a_i - \hat{z}_{\alpha\alpha}]\delta_{\alpha\gamma} - \hat{z}_{\alpha\gamma})\delta_{ij} - \frac{(\beta J)^2}{8} p(p-1)(p-2)\{4q^{p-2} z_{\alpha\gamma} + (p-3) x_\alpha x_\gamma q^{p-4}\} \frac{1}{N} m_i m_j \,. \tag{B6}$$

The last term is of order $O(1/N)$ and corresponds to the two terms under discussion. Therefore they will be dropped and we get from Eq. (B3) after performing a Hubbard–Stratonovic transformation

$$\langle \det \underline{A} \rangle = \int \prod_{\alpha\gamma} \frac{dz_{\alpha\gamma}}{\sqrt{2\pi\lambda N^{-1}}} \int \prod_{i,\alpha} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \sum_{i,\alpha} a_i \xi_{i\alpha}^2 - N \sum_{\alpha,\gamma} \frac{z_{\alpha\gamma}^2}{2\lambda} + \frac{1}{2} \sum_{\alpha,\gamma} \sum_{i} z_{\alpha\gamma} \xi_{i\alpha} \xi_{i\gamma} \right\}, \tag{B7}$$

where $\lambda = (\beta J)^2 p(p-1) q^{p-2}$. We calculate the integral over $z_{\alpha\gamma}$ by steepest descent and choose the diagonal saddle point $z_{\alpha\gamma} = z\delta_{\alpha\gamma}$. We calculated $\langle \det \underline{A} \rangle$ also with the help of Grassman variables instead of replicas (see Ref. 6) and checked in this way the correctness of this saddle point. Then we obtain

$$\begin{aligned} \langle \det \underline{A} \rangle &= c \int \prod_{i,\alpha} \frac{d\xi_i^\alpha}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \sum_{i,\alpha} (a_i - 2z) \xi_{i\alpha}^2 - N \sum_{\alpha} \frac{z^2}{2\lambda} \right\} \\ &= c \prod_i (a_i - 2z)^{-n/2} \exp\left\{ -\frac{Nnz^2}{2\lambda} \right\} \\ &\overset{n \to -2}{\longrightarrow} c \prod_i (a_i - 2z) \exp\left\{ \frac{Nz^2}{\lambda} \right\}, \end{aligned} \tag{B8}$$

where c is a prefactor and z has to be determined variationally.

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