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Application of a minimum-cost flow algorithm to the three-dimensional gauge-glass model with screening

Jens Kisker
Institut für Theoretische Physik, Universität zu Köln, D-50937 Köln, Germany

Heiko Rieger
HLRZ, Forschungszentrum Jülich, D-52425 Jülich, Germany

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We study the three-dimensional gauge glass model in the limit of strong screening by using a minimum-cost flow algorithm, enabling us to obtain *exact* ground states for systems of linear size $L \leqslant 48$. By calculating the domain-wall energy, we obtain the stiffness exponent $\theta = -0.95 \pm 0.03$, indicating the absence of a finite temperature phase transition, and the thermal exponent $\nu = 1.05 \pm 0.03$. We discuss the sensitivity of the ground state with respect to small perturbations of the disorder and determine the overlap length, which is characterized by the chaos exponent $\zeta = 3.9 \pm 0.2$, implying strong chaos. [S0163-1829(98)51138-1]

In high-temperature superconductors, which are strongly type II, disorder plays an important role as a pinning mechanism for vortices. Without disorder, as has been suggested by Abrikosov within mean-field theory, flux lines form a triangular lattice. However, under the influence of an external current perpendicular to the field, the vortices will move since they experience a Lorentz force, and thus energy is dissipated destroying the superconducting state. It has been suggested[1,2] that disorder in the form of point defects may pin the vortices at random positions leading to a superconducting vortex glass phase, where the phase of the order parameter is random in space, but frozen in time, similar to spin glasses.

A simplified model commonly used to study the vortex glass is the gauge glass, which is believed to be in the same universality class and contains the necessary prerequisites, disorder and frustration, for glassy behavior. Within this model, one neglects fluctuations in the amplitude of the order parameter and only considers the phase of the condensate. The Hamiltonian is given by

$$H = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j - A_{ij} - \lambda_0^{-1} a_{ij}) + \frac{1}{2} \sum_{\square} (\nabla \times \mathbf{a})^2, \tag{1}$$

where ϕ_i is the phase of the order parameter on site i and J is the interaction strength, henceforth set to $J=1$. The sum is over all pairs $\langle i,j \rangle$ of nearest neighbors on a simple cubic lattice of size $N = L^3$. The quenched random vector potentials A_{ij} are drawn uniformly from the interval $[0,2\pi]$ and represent the effect of disorder and an external magnetic field. Screening of the interactions between vortices is incorporated by the fluctuating vector potentials a_{ij} which are integrated over from $-\infty$ to ∞ under the gauge constraint $\nabla \cdot \mathbf{a} = 0$, and λ_0 denotes the screening length. The limit $\lambda_0 \to 0$ corresponds to strong screening, whereas $\lambda_0 \to \infty$ is the limiting case without screening. The last term describes the magnetic energy, and is the sum over all plaquettes of the lattice, where the curl is given as the directed sum of the a_{ij} around one plaquette.

Most of the theoretical work so far has concentrated on establishing numerically the lower critical dimension of the gauge glass model, both with and without screening of the interactions between vortices. Without screening, there is no finite temperature transition to a vortex glass phase in two dimensions,[3–6] whereas in three dimensions there is evidence for a finite T_c, as has been found by domain wall renormalization group analyses (DWRG)[7,8] and finite temperature Monte Carlo simulations,[9,10] though due to limited system sizes and insufficient statistics the earlier DWRG studies[4,5,7] could not fully rule out that the lower critical dimension is exactly $d=3$.

Sufficiently close to the critical point, screening effects become important, since the correlation length ξ diverges more strongly than the screening length λ and the two length scales eventually become comparable.[5] The effect of screening was investigated in Ref. 5 by a DWRG study and more recently in Ref. 11 by means of a finite temperature Monte Carlo simulation, and the results indicate that screening is a relevant perturbation, destroying the finite temperature transition in three dimensions, though the DWRG analysis could only be performed for rather small system sizes ($L \leqslant 4$).

In the present paper we reinvestigate the gauge glass model in the limit of strong screening by performing a DWRG analysis using *exact* ground states, which we obtain via a minimum cost flow algorithm from combinatorial optimization. This algorithm allows us to study systems with linear size up to $L=48$, which is considerably larger than the system sizes in the previous studies.[5,11] In addition, we study the sensitivity of the ground state configurations with respect to small parameter changes, thereby obtaining the *chaos* exponent.

We make use of the vortex representation of the gauge glass model, which is obtained from the Hamiltonian in Eq. (1) by making the Villain approximation,[12–14] which replaces the exponentiated cosine term in the partition function by a sum of periodic Gaussians, and then integrating out the spin wave degrees of freedom. Thereby one obtains the vortex Hamiltonian[3]

$$H_V=-\frac{1}{2}\sum_{i,j}(\mathbf{J}_i-\mathbf{b}_i)G(i-j)(\mathbf{J}_j-\mathbf{b}_j), \tag{2}$$

defined on the *dual* lattice, which again is a simple cubic lattice. The $\mathbf{J}_i$ are three-component integer variables running from $-\infty$ to ∞ living on the links of the dual lattice and satisfy the divergence constraint $(\nabla\cdot\mathbf{J})_i=0$ on every site i. The $\mathbf{b}_i$ are magnetic fields which are constructed from the quenched vector potentials A_{ij} by a lattice curl, i.e., one obtains $\mathbf{b}_i$ as $1/(2\pi)$ times the directed sum of the vector potentials on the plaquette surrounding the link on the dual lattice $\mathbf{b}_i$ lives on. By definition, the magnetic fields satisfy the divergence free condition $(\nabla\cdot\mathbf{b})_i=0$ on every site, since they stem from a lattice curl. The vortex interaction is given by the lattice Green's function

$$G(i,j)=J\frac{(2\pi)^2}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}}\frac{1-\exp[i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)]}{2\sum_{n=1}^{d}[1-\cos(k_n)]+\lambda_0^{-2}}. \tag{3}$$

In the case we are interested in, the strong screening limit $\lambda_0\to 0$, $G(i-j)$ reduces to $G(0)=0$ for $i=j$ and $G(i,j)=J(2\pi\lambda_0)^2$ for $i\neq j$ with exponentially small corrections.[11] Thus if we subtract $J(2\pi\lambda_0)^2$ from the interaction and measure the energy in units of $J(2\pi\lambda_0)^2$, one obtains the simpler Hamiltonian

$$H_V=\frac{1}{2}\sum_i(\mathbf{J}_i-\mathbf{b}_i)^2. \tag{4}$$

We remark that H_V is not trivial due to the divergence condition $(\nabla\cdot\mathbf{J})_i=0$.

Finding the ground state of the Hamiltonian in Eq. (4) subject to the constraint $(\nabla\cdot\mathbf{J})_i=0$ can be interpreted as a *minimum-cost flow problem* in the language of combinatorial optimization. From this point of view, the problem can be restated as

$$\text{Minimize } z(\mathbf{J})=\sum_i c_i(\mathbf{J}_i) \tag{5}$$

subject to the constraint $(\nabla\cdot\mathbf{J})_i=0$, where the cost functions $c_i(\mathbf{J}_i)=(\mathbf{J}_i-\mathbf{b}_i)^2/2$ have been defined.

The algorithm we shall use is the *successive shortest path algorithm*,[15–17] which solves the problem in polynomial time in this specific case. For the implementation we made use of the LEDA (Library of efficient data types and algorithms) programming library.[18] We were able to obtain *exact* ground states for system sizes up to $L=48$ on ordinary workstations. The computation time increases approximately with N^2, and one instance for $L=48$ took about 2.5 hours computer time on a Sun Ultra2 (167 MHz) workstation. We remark that a similar algorithm has already been used to calculate ground states of the solid-on-solid model for a surface on a disordered substrate, which also can be mapped on a minimum-cost flow problem.[17]

We now present our results. First we address the question if the gauge glass model in the limit of strong screening shows a finite temperature transition as presumably is the case without screening, i.e., we investigate the stability of the ground state with respect to thermal fluctuations. We make use of the concept of domain wall renormalization, which has been applied to spin glasses in the same context.[19] The idea is to study the energy ΔE necessary to flip a cluster on length scale L. For long length scales L, one expects that

$$\Delta E\sim L^{\theta}, \tag{6}$$

where θ is the stiffness exponent. The sign of θ now determines whether there is a finite temperature phase transition. If θ is positive, then the domain wall energy is increasing with cluster size, and one concludes that the ground state is stable with respect to thermal fluctuations. Correspondingly, if θ is negative, the argument is that the ground state is unstable, since large cluster can be flipped by arbitrarily small energy. Thus in this case, the transition temperature is zero.

The usual way to determine the defect or domain wall energy is to measure the energy difference between ground states obtained for periodic and antiperiodic boundary conditions (bc), respectively. For the model under consideration in the vortex representation (4), however, one has to incorporate a boundary term to mimic the effect of a change in bc.[3,5] This boundary term is rather inconvenient to implement in the minimum-cost flow algorithm we are using, so we propose an alternative method to induce a domain wall or elementary excitation. Note that the vortex Hamiltonian in Eq. (4) with periodic bc and without the additional boundary term corresponds to fluctuating boundary conditions in the gauge glass model.[20,21]

In the gauge glass one changes from periodic to antiperiodic bc by adding π/L to the vector potentials in one spatial direction, e.g., $A_{ij}^x\to A_{ij}^x+\pi/L$. Such a shift of the vector potential, however, has no effect on the Hamiltonian in the vortex representation (4), since the magnetic fields $\mathbf{b}_i$ are constructed from the vector potential by a lattice curl and thus remain unchanged. Only the argument of the boundary term changes, which can be compensated by forming one or several vortex loops with total area[3,5] $L^2/2$. From this observation we derive our basic idea: First we consider the vortex Hamiltonian Eq. (4) with periodic bc and calculate the exact ground state configuration $\{\mathbf{J}^0\}$. The energy $E_0(\{\mathbf{J}^0\})$ of this state is obtained via H_V in Eq. (4). We then determine the global flux f of this configuration in one spatial direction, e.g., in the x direction

$$f_x=\frac{1}{L}\sum_i \mathbf{J}_i^x, \tag{7}$$

where f_x can be interpreted as a total winding number. Next, we gradually decrease all costs for a flow increment in the x direction [given by $c_{ix}(\mathbf{J}_i+1)-c_{ix}(\mathbf{J}_i)$] simultaneously, which makes global flux in this direction energetically more favorable, whereas the costs for all topologically simply connected loops (those with winding number zero around the $3d$ torus) remain unchanged. We reduce the costs until we obtain a configuration $\{\mathbf{J}^1\}$ with global flux f_x+1, which is an elementary low energy excitation with length scale L. We calculate the energy E_1 of this configuration again with H_V in Eq. (4) with the original cost functions in Eq. (5) [i.e., $E_1=\sum_i c_i(\mathbf{J}_i^1)$]. The domain wall energy is then given by $\Delta E=E_1-E_0$, which is always positive since the new state $\{\mathbf{J}^1\}$ with flux f_x+1 corresponds to an excited state.

In a small fraction ($\approx 5\%$) of the samples, the flux changes discontinuously by more than one unit upon slowly

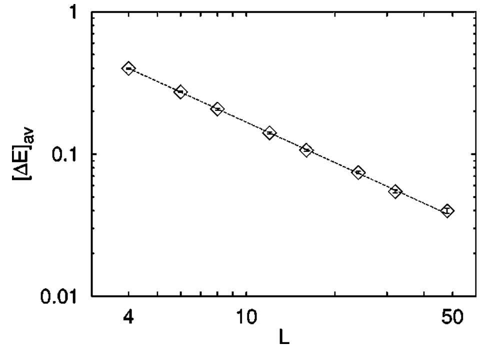

FIG. 1. The domain wall energy $[\Delta E]_{av}$ in a log-log plot. The straight line is a fit to $[\Delta E]_{av} \sim L^{\theta}$ with $\theta = -0.95 \pm 0.03$. This implies a thermal exponent of $\nu = 1.05 \pm 0.03$. The disorder average is over 500 samples for $L = 48$, 1500 samples for $L = 32$, and for the smaller sizes several thousand samples have been used.

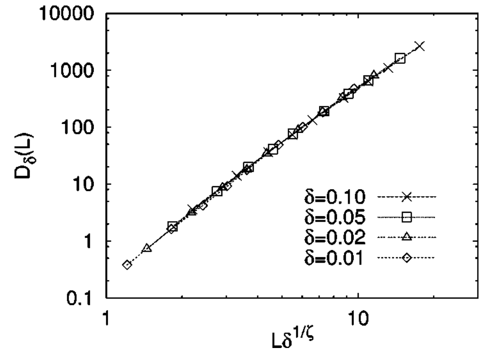

FIG. 2. Scaling plot of the overlap function $D_\delta(L)$. The data scale nicely with the scaling variable $L\delta^{1/\zeta}$. The chaos exponent is given by $\zeta = 3.9 \pm 0.2$. The system sizes studied range from $L = 4$ to $L = 32$, where 5000 samples have been used for $L \leq 20$, 2000 samples for $L = 24$ and 500 samples for $L = 32$.

decreasing the costs as described above. However, the resulting configuration still represents an elementary excitation of length scale L and we can also use this configuration for calculating the energy.

We note that it is easy to see that in the vortex (or Villain) representation of the pure classical three-dimensional XY model which is given by $H = \Sigma_i \mathbf{J}_i^2$, the elementary low energy excitation is indeed given by a configuration with one additional flux line of length L winding once around the $3d$ torus. The energy difference between the ground state ($f_x = 0$) and the excited state ($f_x = 1$) then simply is $\Delta E \sim L$, which corresponds to the energy of a spin wave excitation with minimum wave vector in the $3d$ XY model in the phase representation.

Figure 1 shows the domain wall energy ΔE vs L for $L \leq 48$ in a double logarithmic plot. One observes a straight line behavior which can be nicely fitted by

$$\Delta E \sim L^{\theta} \quad \text{with} \quad \theta = -0.95 \pm 0.03. \tag{8}$$

Thus we reestablish that $T_c = 0$ for the gauge glass model in the strong screening limit, as has been found in Refs. 5 and 11. From the stiffness exponent θ the thermal exponent ν, which describes the divergence of the correlation length, can be calculated. For $T_c = 0$ it is $\xi \sim T^{-\nu}$ and by equating the thermal energy with the energy of a low lying excitation on the length scale of the correlation length, it follows that

$$\nu = \frac{1}{|\theta|}. \tag{9}$$

From this relation we obtain $\nu = 1.05 \pm 0.03$, which agrees well with a result from a finite temperature Monte Carlo simulation for the same model by Wengel and Young,[11] who were able to study system sizes $L \leq 12$ and found a zero temperature phase transition with $\nu = 1.05 \pm 0.1$.

Next we want to discuss the issue of *chaos* in the gauge glass model. From spin glasses it is known, that infinitesimal changes of parameters like the temperature or the couplings can have quite a dramatic effect on the ground state or equilibrium configuration.[19,22] It is argued that a so-called overlap length L^* exists, which is a measure for the length scale, up to which the domain structure essentially remains unchanged after an infinitesimal perturbation of a parameter. For length scales larger than the overlap length, the domain structure changes. For spin glasses, the overlap length is given by[19]

$$L^* \sim \delta^{-1/\zeta} \quad \text{with} \quad \zeta = \frac{d_s}{2} - \theta, \tag{10}$$

where the result is obtained by equating the energy of a droplet excitation L^{θ} and the energy change caused by the perturbation which is proportional to $L^{d_s/2}$, where d_s is the fractal dimension of the droplets.

To pursue a similar investigation in the gauge glass model we study the resulting changes in the ground state configuration when the random vector potentials A_{ij} are perturbed by a small amount. To be specific, we define new vector potentials by $A'_{ij} = A_{ij} + \epsilon_{ij}$, where ϵ_{ij} is randomly drawn from the interval $[-\delta, \delta]$. We calculate the ground state for both realizations of the disorder $\{A_{ij}\}$ and $\{A'_{ij}\}$, and define the distance D between the resulting ground state configurations $\{\mathbf{J}\}$ and $\{\mathbf{J}'\}$ by

$$D_\delta = \sum_i (\mathbf{J}_i - \mathbf{J}'_i)^2. \tag{11}$$

Note that for small values of $D_\delta(L)$ the ground states are more correlated than for larger values.

To determine the chaos exponent, we perform a scaling plot of the data for the overlap function $D_\delta(L)$. Guided by Eq. (10), we attempt a scaling plot with the scaling variable $L\delta^{1/\zeta}$, where ζ is the chaos exponent. Such a plot is shown in Fig. 2, and one observes that the data scale nicely with $\zeta = 3.9 \pm 0.2$. This relatively large value of ζ implies *strong* chaos, since in the limit of a vanishing perturbation $\delta \to 0$, the overlap length, which is proportional to $\delta^{-1/\zeta}$, increases slowly. The value of $\zeta = 3.9 \pm 0.2$ is considerably larger than for instance the one for the two-dimensional Ising spin glass, where $\zeta = 0.95 \pm 0.05$ has been obtained.[22,23]

Chaos cannot only be observed with respect to perturbations of the disorder, but also with respect to small temperature changes.[19] For a continuous bond distribution and low temperatures, the overlap length is expected[19] to behave as $L^*_{th} \sim T^{-2/\zeta}$ for $T_c = 0$. The other relevant length scale is the correlation length $\xi \sim \delta^{-\nu}$, and for the $2d$ Ising spin glass with $\nu\approx 2$ and $2/\zeta\approx 2$, the exponents characterizing the two length scales L^*_{th} and ξ appear to be equal,[22–24] so one might speculate that they are related. However for the gauge glass model we find $2/\zeta\approx 0.5$ and $\nu\approx 1$, i.e., the divergence of the thermal correlation length is much stronger. It would be interesting to study chaos with respect to temperature changes in the gauge glass explicitly.

Summarizing, we studied the strongly screened gauge glass model in the vortex representation. Using a polynomial time minimum-cost flow algorithm we could deal with much larger system sizes than considered before in the literature. We calculated the *exact* ground states and also configurations with a (global) low energy excitation. In this way we could perform a finite size scaling analysis of the so-called domain wall energy and obtained a pretty accurate estimate for the stiffness exponent, that is, $\theta=-0.95\pm 0.03$. From this we can draw two conclusions: (a) since it is clearly negative, there is no superconducting glass phase (or vortex glass phase) at nonvanishing temperature, and (b) the thermal correlation length diverges only with $T\to 0$ as $\xi_{\rm th}\sim T^{-\nu}$ with an exponent $\nu\sim 1.05\pm 0.03$, which is in agreement with Refs. 5 and 11. Finally, we studied the effect of a small perturbation of the disorder on the ground state domain structure and found that the overlap length is characterized by the chaos exponent $\zeta=3.9\pm 0.2$. This is a pretty large value implying a fast destruction of ground state correlations by thermal fluctuations.

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[1] M. P. A. Fisher, Phys. Rev. Lett. **62**, 1415 (1989).

[2] D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B **43**, 130 (1991).

[3] M. P. A. Fisher, T. A. Tokuyasu, and A. P. Young, Phys. Rev. Lett. **66**, 2931 (1991).

[4] M. J. P. Gingras, Phys. Rev. B **45**, 7547 (1992).

[5] H. S. Bokil and A. P. Young, Phys. Rev. Lett. **74**, 3021 (1995).

[6] J. M. Kosterlitz and M. V. Simkin, Phys. Rev. Lett. **79**, 1098 (1997).

[7] J. D. Reger, T. A. Tokuyasu, A. P. Young, and M. P. A. Fisher, Phys. Rev. B **44**, 7147 (1991).

[8] J. Maucourt and D. R. Grempel, Phys. Rev. B **58**, 2654 (1998).

[9] D. A. Huse and H. S. Seung, Phys. Rev. B **42**, 1059 (1990).

[10] C. Wengel and A. P. Young, Phys. Rev. B **56**, 5918 (1997).

[11] C. Wengel and A. P. Young, Phys. Rev. B **54**, R6869 (1996).

[12] J. Villain, J. Phys. (France) **36**, 581 (1975).

[13] J. V. José, L. P. Kadanoff, S. K. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[14] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989).

[15] R. Ahuja, T. Magnanti, and J. Orlin, *Network Flows* (Prentice Hall, New Jersey, 1993).

[16] H. Rieger, *Frustrated Systems: Ground State Properties via Combinatorial Optimization*, Lecture Notes in Physics Vol. 501 (Springer- Verlag, Heidelberg, 1998); cond-mat/9807210.

[17] H. Rieger and U. Blasum, Phys. Rev. B **55**, R7394 (1997); U. Blasum, W. Hochstättler, U. Moll, and H. Rieger, J. Phys. A **29**, L459 (1996).

[18] Information on the LEDA (Library of efficient data types and algorithms) library can be found at http://www.mpi-sb.mpg.de/LEDA/leda.html.

[19] A. J. Bray and M. A. Moore, Phys. Rev. Lett. **58**, 57 (1987).

[20] P. Olsson, Phys. Rev. B **52**, 4511 (1995).

[21] P. G. Gupta, S. Teitel, and M. J. P. Gingras, Phys. Rev. Lett. **80**, 105 (1998).

[22] H. Rieger, L. Santen, U. Blasum, M. Diehl, and M. Jünger, J. Phys. A **29**, 3939 (1996).

[23] M. Ney-Nifle and A. P. Young, J. Phys. A **30**, 5311 (1997).

[24] N. Kawashima, N. Hatano, and M. Suzuki, J. Phys. A **25**, 4985 (1992).