Anomalous diffusion in aperiodic environments

Ferenc Iglói*
*Research Institute for Solid State Physics and Optics, P.O. Box 49, H-1525 Budapest, Hungary
and Laboratoire de Physique des Matériaux, Université Henri Poincaré (Nancy 1), F-54506 Vandœuvre lès Nancy, France*

Loïc Turban
Laboratoire de Physique des Matériaux, Université Henri Poincaré (Nancy 1), F-54506 Vandœuvre lès Nancy, France

Heiko Rieger
Höchstleistungsrechenzentrum, Forschungszentrum Jülich, 52425 Jülich, Germany
(Received 8 September 1998)

We study the Brownian motion of a classical particle in one-dimensional inhomogeneous environments where the transition probabilities follow quasiperiodic or aperiodic distributions. Exploiting an exact correspondence with the transverse-field Ising model with inhomogeneous couplings, we obtain many analytical results for the random walk problem. In the absence of global bias the qualitative behavior of the diffusive motion of the particle and the corresponding persistence probability strongly depend on the fluctuation properties of the environment. In environments with bounded fluctuations the particle shows normal diffusive motion and the diffusion constant is simply related to the persistence probability. On the other hand, in a medium with unbounded fluctuations the diffusion is ultraslow and the displacement of the particle grows on logarithmic time scales. For the borderline situation with marginal fluctuations both the diffusion exponent and the persistence exponent are continuously varying functions of the aperiodicity. Extensions of the results to disordered media and to higher dimensions are also discussed. [S1063-651X(99)04402-5]

PACS number(s): 64.60.−i, 66.30.Dn, 75.10.Nr, 05.50.+q

I. INTRODUCTION

Brownian motion is perhaps the best understood stochastic process in classical physics in both homogeneous environments and disordered media. The study of the diffusion problem in inhomogeneous environments is physically motivated by transport processes (molecular diffusion, flow lines in a porous medium, and electrical conduction) on the one hand and the relaxational properties of disordered systems (random magnets and spin glasses) on the other (see Refs. [1,2]).

In the presence of asymmetric transition rates, i.e., when the probability per unit time $w_{\mathbf{r},\mathbf{r}'}$ for a particle to jump from site $\mathbf{r}$ to site $\mathbf{r}'$ is different from $w_{\mathbf{r}',\mathbf{r}}$, the disorder strongly modifies the behavior of the diffusive motion in $d<2$ dimensions. In one space dimension, where the effect of disorder is most pronounced, the diffusion is ultraslow and the averaged mean-square displacement grows on a logarithmic time scale [3]:

$$[\langle X^2(t)\rangle]_{av}\sim \ln^4 t. \tag{1.1}$$

Another type of inhomogeneity is provided by fractal lattices, either regular or random, such as percolation clusters, in which the Brownian motion has been intensively studied under the name ''the ant in the labyrinth'' [4,5,2]. Also much work has been devoted to the clarification of diffusion processes in the presence of hierarchically distributed energy barriers [5,6], a problem that is related to relaxation processes in disordered systems [7].

In the present work we study the Brownian motion in inhomogeneous environments where the transition rates are asymmetric and distributed according to quasiperiodic or, more generally, aperiodic rules. As a related earlier work, one may mention an investigation of the Brownian motion on the two-dimensional Penrose lattice, where normal diffusive behavior has been found [8]. Here we mainly concentrate on one-dimensional aperiodic systems. Besides its mathematical interest, the present study is also physically motivated since artificial multilayer systems with controlled distributions of the atomic layers may now be grown by molecular beam epitaxy [9]. When particle transport has different time scales for the motion parallel and perpendicular to the layers, respectively, a one-dimensional diffusion process perpendicular to the layers is, in principle, observable.

The study of cooperative phenomena in quasiperiodic and aperiodic systems is an intensive field of research. One may mention phase transitions and critical phenomena in Ising and other magnetic models, percolation, self-avoiding walks, etc. Aperiodic structures, which interpolate between periodic and random systems, may or may not influence qualitatively the properties of a cooperative process. Concerning the critical behavior of aperiodic magnetic systems, a relevance-irrelevance criterion has been proposed [10,11], which is an extension of the well-known Harris criterion for disordered systems [12]. The vast amount of exact results about the critical properties of aperiodic quantum Ising chains [13–17] and related aperiodically layered two-dimensional Ising models [18] are all in accordance with this criterion. Aperi-

*Permanent address: Institute for Theoretical Physics, Szeged University, H-6720 Szeged, Hungary.

odicity may also change to second order a transition that is of first order in the pure system, as was demonstrated recently in a numerical study [19].

In the present work we show that a relevance-irrelevance criterion, similar to that of magnetic systems, can be formulated for the Brownian motion, which is then checked against exact results obtained for different one-dimensional aperiodic environments.

The paper is organized as follows. In Sec. II we introduce the basic notations and quantities (drift velocity, diffusion constant, and persistence probability) and present the relevance-irrelevance criterion for one-dimensional aperiodic environments. In Sec. III an exact correspondence between the random walk (RW) and the transverse-field Ising model (TIM) is presented in one dimension, which is then used in Sec. IV to obtain analytical results for irrelevant, relevant, and marginal aperiodic environments. Our results are extended to higher dimensions in Sec. V and discussed in Sec. VI.

II. FORMALISM AND THE RELEVANCE-IRRELEVANCE CRITERION

We consider a one-dimensional RW with nearest neighbor hopping, characterized by transition probabilities $w_{i,i\pm 1}$ for a jump from site i to site $i\pm 1$. The time evolution of $P_i(t)$, the probability for the particle to be on site i at time t, is governed by the master equation

$$\frac{dP_i}{dt}=w_{i-1,i}P_{i-1}-(w_{i,i-1}+w_{i,i+1})P_i+w_{i+1,i}P_{i+1}. \tag{2.1}$$

The transition probabilities are generally nonsymmetric. Here we suppose that their ratio is given by

$$\frac{w_{i,i+1}}{w_{i+1,i}}=\epsilon_i=\epsilon R^{f_i}, \tag{2.2}$$

where $R>0$ is the amplitude of the inhomogeneity ($R=1$ corresponds to the homogeneous lattice) and the integers f_i are taken from an aperiodic or a quasiperiodic sequence. For the sake of simplicity in the following we take $w_{i+1,i}=w_{\leftarrow}=1$.

The aperiodic chain may be replaced by a periodic approximant of period N such that $\epsilon_i=\epsilon_{i+N}$ [20,10]. The aperiodic system is recovered in the limit $N\to\infty$. Exact expressions for the drift velocity v_d and the diffusion constant D obtained by Derrida for an arbitrary distribution of the transition rates [20] can then be used to treat the aperiodic case.

With our notations, the drift velocity of the aperiodic model may be written as

$$v_d=\frac{1-\prod_{n=1}^{N}\epsilon_n^{-1}}{1-\prod_{n=1}^{N}\epsilon_n^{-1}+\frac{1}{N}\sum_{n=1}^{N}\sum_{i=1}^{N}\epsilon_{n+1}^{-1}\epsilon_{n+2}^{-1}\cdots\epsilon_{n+i}^{-1}}. \tag{2.3}$$

Defining the control parameter as

$$\delta=\frac{1}{N}\sum_{n=1}^{N}\ln\epsilon_n, \tag{2.4}$$

the drift velocity is zero for $\delta=0$, whereas for a small bias such that $\delta N\ll 1$, the drift velocity is proportional to δ:

$$v_d=\delta D_0,\quad \frac{1}{D_0}=\frac{1}{N^2}\sum_{n=1}^{N}\sum_{i=1}^{N}\epsilon_{n+1}^{-1}\epsilon_{n+2}^{-1}\cdots\epsilon_{n+i}^{-1}. \tag{2.5}$$

One can similarly calculate the diffusion constant D, which in the zero bias case $\delta=0$ is simply given by

$$D(\delta=0)=D_0. \tag{2.6}$$

Before analyzing the diffusive behavior of aperiodic walks we consider another quantity of interest, the persistence probability $P_{per}(L,t)$, which is the probability that the walker starting at site $i=1$ does not cross its starting position until time t. Here the length scale L in the definition is set by the presence of an adsorbing site at $i=L+1$, thus $w_{L+1,L}=0$. Due to this adsorbing site the persistence probability has a finite long time limit $\lim_{t\to\infty}P_{per}(L,t)=p_{per}(L)$, which can be expressed as [21]

$$p_{per}(L)=\left(1+\sum_{i=1}^{L}\prod_{j=1}^{i}\epsilon_j^{-1}\right)^{-1}. \tag{2.7}$$

It is easy to see that, in the thermodynamic limit, $\lim_{L\to\infty}p_{per}(L)=p_{per}$ plays the role of an order parameter: It is nonvanishing for $\delta>0$ only. For the homogeneous system with $R=1$ in Eq. (2.2)

$$p_{per}^{hom}=1-\epsilon^{-1}\simeq\delta, \tag{2.8}$$

whereas at the critical point in a finite homogeneous system

$$p_{per}^{hom}(L,\delta=0)=\frac{1}{L+1}. \tag{2.9}$$

$P_{per}(t)=\lim_{L\to\infty}P_{per}(L,t)$ is the usual persistence probability for an infinite system.

Next we analyze the expressions of the basic quantities in Eqs. (2.5) and (2.7). The qualitative behavior for any *periodic system*, i.e., when N is finite, is the same in the vicinity of the critical point: The diffusion constant D_0 in Eq. (2.5) is finite and the persistence probability in Eq. (2.7) has a linear δ or L^{-1} dependence.

For aperiodic systems, which are obtained in the limit $N\to\infty$, one has to consider the products of the transition rates $\mathcal{P}_n(L)=\prod_{i=1}^{L}\epsilon_{n+i}=\exp[\Delta_n(L)]$, where $\Delta_n(L)=\sum_{i=1}^{L}\ln\epsilon_{n+i}$ measures the fluctuations in the ''energy landscape'' when $\ln\epsilon_i$ is related to the height of the ith energy barrier in an activated diffusion process. Here we differentiate between three possibilities.

(i) For *bounded fluctuations* in the transition rates, i.e., with a flat energy landscape, the products $\mathcal{P}_n(L)$ stay finite at the critical point, thus the diffusion constant D_0 remains finite and the linear δ dependence of the persistence probability is maintained. Thus this type of aperiodicity is *irrelevant* and the aperiodic Brownian motion keeps the same critical properties as the homogeneous one.

(ii) For *unbounded fluctuations* in the transition rates, when the energy landscape is rough, so that $\Delta_n(L) \sim L^{\Omega}$ with a wandering exponent $\Omega > 0$, the products $\mathcal{P}_n(L)$ that appear in Eqs. (2.5) and (2.7) are divergent at the critical point when $L \to \infty$. Consequently, the diffusion constant D_0 vanishes and the persistence probability displays a nonlinear δ dependence. We conclude that an aperiodic environment with unbounded fluctuations is a *relevant perturbation* for the Brownian motion. As we show later, diffusion in a relevant aperiodic environment is *ultraslow*, i.e., the mean-square displacement grows on a logarithmic time scale, as for the Sinai model in Eq. (1.1).

(iii) In the borderline case, the wandering exponent of the environment is $\Omega = 0$, thus fluctuations in the energy landscape grow logarithmically. In this *marginal situation* the products of the transition rates $\mathcal{P}_n(L)$ have a power law L dependence and the different physical quantities also display power law singularities. For this type of *anomalous diffusion* the drift velocity depends on the bias as

$$v_d \sim \delta^{\tau}, \quad |\delta| \ll 1, \tag{2.10}$$

with $\tau > 1$, whereas the mean square displacement grows like

$$\langle X^2(t) \rangle \sim t^{\psi}, \tag{2.11}$$

with $\psi < 1$. Finally, the persistence probability is in general characterized by algebraic singularities

$$p_{per} \sim \delta^{\chi}, \quad P_{per}(t) \sim t^{-\Theta}. \tag{2.12}$$

The critical exponents defined above are not independent. Using results of Sec. III in Eqs. (3.7) and (3.9), one can show that the relevant time scale t and the bias δ are related by $t^{-1} \sim v_d \delta \sim \delta^{1+\tau}$. For the length scale, one has $L \sim v_d t \sim \delta^{-1}$. These relations lead to the scaling laws

$$\tau = \frac{2}{\psi} - 1, \quad \chi = \frac{2}{\psi}\Theta, \tag{2.13}$$

which evidently hold for normal diffusion with $\tau = \psi = \chi = 1$ and $\Theta = 1/2$. Thus two critical exponents are enough to describe the behavior of the anomalous diffusion.

To obtain the characteristic quantities for relevant and marginal environments one has to solve the master equation (2.1), which amounts to solving the eigenvalue problem for the transition matrix or Focker-Planck (FP) operator

$$\underline{\underline{M}}\,\underline{v}_k = \lambda_k \underline{v}_k, \quad \underline{u}_k^T \underline{\underline{M}} = \underline{u}_k^T \lambda_k, \tag{2.14}$$

where the matrix elements take the form $(\underline{\underline{M}})_{i,j} = w_{i,j}$ for $i \neq j$ and $(\underline{\underline{M}})_{i,i} = -\Sigma_j w_{i,j}$. All the physical properties of the model can be expressed in terms of the left and right eigenvectors $\underline{u}_k$ and $\underline{v}_k$, respectively, and the eigenvalues λ_k, which are nonpositive. For example, the probability $P_{i,j}(t)$ that the walker starting at $t=0$ at site i arrives on site j at time t is given by

$$P_{i,j}(t) = \sum_k u_k(i) v_k(j) \exp(\lambda_k t). \tag{2.15}$$

The relevant time scale of the problem is related to the inverse of the largest nonzero eigenvalues and the dynamical properties of the RW are connected to the scaling behavior of the eigenvalues of the FP operator at the top of the spectrum. Considering a large finite system of size L, under a change of the length scale by a factor of $b > 1$, such that $L' = L/b$, the eigenvalues at the top of the spectrum with $k \ll L$ are expected to transform as

$$\lambda_k' = b^{y_\lambda} \lambda_k, \tag{2.16}$$

where y_λ is the scaling dimension of the gap. From Eq. (2.16) the finite-size behavior of the eigenvalues $\lambda_k(L) \sim L^{-y_\lambda}$ follows, thus the time and length scales are related through $t \sim L^{y_\lambda}$. On the other hand, from Eq. (2.11) $L^2 \sim t^{\psi}$, thus the diffusion exponent is given by

$$\psi = \frac{2}{y_\lambda}. \tag{2.17}$$

For relevant aperiodic environments, the leading eigenvalues have a stretched exponential finite-size dependence, thus the diffusion exponent is formally zero.

In the master equation formalism, the persistence probability $P_{per}(L,t)$ can be calculated by putting adsorbing sites at $i=0$ and $i=L+1$. Then $P_{per}(L,t) = P_{1,L+1}(t)$, which in the large-$t$ limit is just the first component of the zero-mode left eigenvector $p_{per}(L) = u_1(1)$, as given by Eq. (2.7).

III. CORRESPONDENCE BETWEEN THE RANDOM WALK AND THE ISING MODEL IN A TRANSVERSE FIELD

First we rewrite the eigenvalue problem of the FP operator in Eq. (2.14) in terms of the components of the right eigenvector $v_k(i)$ as

$$w_{i-1,i} v_k(i-1) - (w_{i,i-1} + w_{i,i+1}) v_k(i) + w_{i+1,i} v_k(i+1) = \lambda_k v_k(i) \tag{3.1}$$

and consider a finite system of size L, i.e., we set $w_{0,1} = w_{L+1,L} = 0$. Then we introduce the new variables

$$v(i) = \alpha_i \tilde{v}(i),$$

$$\alpha_{i+1} = \alpha_i \left(\frac{w_{i,i+1}}{w_{i+1,i}} \right)^{1/2} = \alpha_1 \left(\prod_{j=1}^{i} \frac{w_{j,j+1}}{w_{j+1,j}} \right)^{1/2}, \tag{3.2}$$

in terms of which the eigenvalue problem is transformed into

$$(w_{i-1,i} w_{i,i-1})^{1/2} \tilde{v}_k(i-1) - (w_{i,i-1} + w_{i,i+1}) \tilde{v}_k(i) + (w_{i+1,i} w_{i,i+1})^{1/2} \tilde{v}_k(i+1) = \lambda_k \tilde{v}_k(i), \tag{3.3}$$

which is a real symmetric eigenvalue problem $\Sigma_j T_{ij} \tilde{v}_k(j) = \lambda \tilde{v}_k(i)$ with $T_{ij} = T_{ji}$. Consequently, the eigenvalues λ_k of the FP operator are real.

Next we show that the eigenvalue problem (3.3) formally appears in the free-fermion representation of the transverse-field Ising spin chain, with couplings J_i and transverse fields h_i, described by the Hamiltonian

$$H=-\sum_{i=1}^{L-1} J_i\sigma_i^x\sigma_{i+1}^x-\sum_{i=1}^{L} h_i\sigma_i^z, \tag{3.4}$$

where σ_i^x and σ_i^z are Pauli matrices at site i. The Hamiltonian H can be transformed into a free-fermion model by standard techniques [22]:

$$H=\sum_k \Lambda_k(\eta_k^\dagger\eta_k-1/2). \tag{3.5}$$

Here the $\eta_k^\dagger$ (η_k) are fermion creation (annihilation) operators and the excitation energy Λ_k is the solution of the eigenvalue equation

$$J_{i-1}h_{i-1}\Phi_k(i-1)+(J_{i-1}^2+h_i^2)\Phi_k(i)+J_ih_i\Phi_k(i+1)=\Lambda_k^2\Phi_k(i). \tag{3.6}$$

Comparing Eqs. (3.6) and (3.3), one can notice that they can be cast into the same form with the correspondences

$$\begin{aligned} J_i&\Leftrightarrow(w_{i+1,i})^{1/2},\\ h_i&\Leftrightarrow(w_{i,i+1})^{1/2},\\ \Phi_k(i)&\Leftrightarrow(-1)^i\tilde{v}_k(i),\\ \Lambda_k^2&\Leftrightarrow-\lambda_k. \end{aligned} \tag{3.7}$$

Thus there is a *mathematical equivalence* between the RW in an inhomogeneous environment and the TIM with the corresponding inhomogeneous couplings, as described in Eq. (3.7). One can, however, go further and show that there are several *physical quantities* that are closely related in the two problems.

First let us consider the persistence probability $p_{per}(L)$, which is calculated with adsorbing boundary conditions $w_{0,1}=w_{L+1,L}=0$ as $p_{per}(L)=u_1(1)v_1(L+1)=\tilde{u}_1(1)\alpha_1^{-1}\alpha_{L+1}\tilde{v}_1(L+1)=[\tilde{v}_1(L+1)]^2$. Now using the correspondences in Eq. (3.7), the equivalent TIM with $h_0=J_L=0$ has a fixed spin at $i=0$, whereas the other end of the chain at $i=L+1$ is free [23]. The surface magnetization of the chain measured on the $i=(L+1)$th spin is given by $\overline{m}_s(L)=\Phi_1(L+1)$ [23,24], thus using Eq. (3.7) we have a relation between the surface magnetization of the TIM and the persistence probability of the RW:

$$[\overline{m}_s(L)]^2\Leftrightarrow p_{per}(L). \tag{3.8}$$

We note that this relation has already been mentioned in Ref. [21].

Let us now analyze the relation between the energy scales in the two problems λ_k and Λ_k, respectively. First one may notice that, according to the last of Eqs. (3.7), the eigenvalues λ_k of the FP operator are nonpositive, as they should on physical grounds.

For the *periodic* TIM with $J_i=h$ and $h_i=h_{i+N}=h\epsilon_i^{1/2}$, the energy of low-lying modes, close to the critical point $\delta=0$, is given by a perturbation calculation [10] as

$$\Lambda_k^2\simeq v_s^2(\delta^2+q_k^2), \tag{3.9}$$

where $q_k\sim 1/N$ denotes the wave number corresponding to the smallest excitation energy. Comparing the expression of the sound velocity v_s in Ref. [10] to D_0 in Eq. (2.5), we obtain another correspondence

$$v_s^2\Leftrightarrow D_0. \tag{3.10}$$

Taking the infinite periodic approximant limit $N\to\infty$, we obtain the following important result: If in an environment the diffusion is *normal*, i.e., the diffusion constant D_0 is finite, then the phase transition of the corresponding TIM with inhomogeneous couplings is in the *Onsager universality class*. On the other hand, if the diffusion is *anomalous*, i.e., the diffusion exponent is $\psi<1$, then the phase transition of the TIM is *not of Onsager type*. In the marginal case the phase transition in the TIM is characterized by an anisotropy exponent (or dynamical exponent) $z>1$, which describes the finite-size scaling behavior of the gap $\Lambda_k\sim L^{-z}$ for $k\ll L$. According to Eq. (3.7), z is related to the diffusion exponent through

$$z\Leftrightarrow\frac{1}{\psi}. \tag{3.11}$$

On the other hand, in the relevant case, Λ_k has a stretched exponential L dependence, thus the dynamical exponent is formally infinite.

Finally, we present a useful estimate for the smallest (in absolute value) nonzero eigenvalue of the FP operator λ_{min} by transforming a related result for the TIM [16,23]. Having a large finite chain of length L with a reflecting boundary condition (BC) at $i=1(w_{1,0}=w_{0,1}=0)$ and adsorbing BC at $i=L+1(w_{L+1,L}=0,\ w_{L,L+1}\neq 0)$, then the leading finite-size behavior of $\lambda_{min}(L)$ is connected to the product of the persistence probabilities at the two ends of the chain as

$$\lambda_{min}(L)\simeq -p_{per}(L)\overline{p}_{per}(L)\prod_{i=1}^{L}\epsilon_i^{-1}, \tag{3.12}$$

where the persistence probability at site L is given by

$$\overline{p}_{per}(L)=\left(1+\sum_{i=1}^{L-1}\prod_{j=1}^{i}\epsilon_{L-j}^{-1}\right)^{-1}, \tag{3.13}$$

whereas $p_{per}(L)$ is given by Eq. (2.7), however, for $L-1$ sites.

IV. DIFFUSION IN APERIODIC ENVIRONMENTS

Here we consider different one-dimensional aperiodic environments and study the corresponding diffusive behavior by analytical methods. As we mentioned before, the relevance of the perturbation is connected to the fluctuation properties of the aperiodic environment.

To be more specific, we consider the behavior at criticality ($\delta=0$) of the logarithm of the transition rates $\ln\epsilon_i$, which is related to the height of the ith energy barrier, Δe_i in an activated diffusion process. When the global bias vanishes, ϵ

takes its critical value ϵ_c such that, according to Eq. (2.4),

$$\epsilon_c = R^{-\rho_\infty}, \quad \rho_\infty = \lim_{L\to\infty} n_L/L, \quad n_L = \sum_{i=1}^{L} f_i, \tag{4.1}$$

where ρ_∞ is the asymptotic density of the perturbed transition probabilities. Then the fluctuation of the energy landscape in a large system of size L is characterized by

$$\Delta(L) = \sum_{i=1}^{L} \ln \epsilon_i = n_L \ln R + L \ln \epsilon_c$$
$$= (n_L - L\rho_\infty) \ln R \sim L^{\Omega} \ln R. \tag{4.2}$$

Here Ω is the wandering exponent of the aperiodic sequence [25], which is easily obtained for aperiodic substitutional sequences.

Working with a finite alphabet $A, B, \ldots$, such that, via subsitution, $A \to S(A)$, $B \to S(B)$, etc., the sequence f_i in Eq. (2.2) is obtained by starting with one of the letters, iterating the substitution process, and finally replacing the letters by digits (or groups of digits) 0 and 1. For a two-letter sequence, one can directly proceed with substitutions on 0 and 1.

The fluctuation properties of a sequence can be deduced from its substitution matrix with entries n_{ij} giving the numbers of letters $i = A, B, \ldots$ in $S(j)(j = A, B, \ldots)$ [25]. The wandering exponent Ω involves the two largest eigenvalues in modulus of this matrix, μ_1 and μ_2, and reads $\Omega = \ln|\mu_2|/\ln \mu_1$.

We now consider different environments associated with specific aperiodic sequences. Environments with bounded $(\Omega<0)$, unbounded $(\Omega>0)$, and marginal $(\Omega=0)$ fluctuations are treated separately. Most of the results can be obtained by adapting the analytical methods developed for the aperiodic TIM, using the correspondences given in Eqs. (3.7), (3.8), (3.10), and (3.11).

A. Aperiodic environments with bounded fluctuations

1. Quasiperiodic (Fibonacci) environment

Quasiperiodic lattices can be generated in several different ways, among others by the well known cut-and-project method. Here we use the following algebraic definition for a one-dimensional quasiperiodic sequence:

$$f_i = 1 + \left[\frac{i}{\omega}\right] - \left[\frac{i+1}{\omega}\right], \tag{4.3}$$

where $[x]$ is the integer part of x and $\omega > 1$ is irrational.

The Fibonacci sequence can be generated by the substitution rule $0 \to 010$, $1 \to 01$ [26] starting with 0. When read from left to right, it corresponds to Eq. (4.3) with $\omega = (\sqrt{5}+1)/2$ the golden mean.

The sequence in Eq. (4.3) and the corresponding quasiperiodic environment, as defined in Eq. (2.2), have bounded fluctuations since the wandering exponent in Eq. (4.2) is $\Omega = -1$. Consequently, the diffusion constant D_0 in Eq. (2.5) is finite. As shown in Appendix A, it can be expressed in closed form using the methods of Ref. [10] as

$$D_0 = \left(\frac{\ln R}{R^{1/2} - R^{-1/2}}\right)^2. \tag{4.4}$$

It is interesting to note that D_0 does not depend on the value of the irrational parameter ω.

The persistence probability in Eq. (2.7) can also be evaluated analytically using the techniques of Ref. [27] (see Appendix A). For small bias there is a linear δ dependence

$$p_{per}(\delta) = R^{1-1/\omega} \frac{\ln R}{R-1} \delta, \quad 0 < \delta \ll 1, \tag{4.5}$$

where the prefactor depends on ω.

For the other end of the system, except for the first digit, which is irrelevant, the sequence, read from right to left, is given by

$$f_i = \left[\frac{i+\omega}{\omega^2}\right] - \left[\frac{i+\omega-1}{\omega^2}\right], \tag{4.6}$$

which leads to the persistence probability

$$\overline{p}_{per}(\delta) = R^{-1+1/\omega} \frac{\ln R}{1-R^{-1}} \delta, \quad 0 < \delta \ll 1. \tag{4.7}$$

One may notice a simple relation between the diffusion constant and the persistence probabilities

$$p_{per}(\delta)\overline{p}_{per}(\delta) = D_0 \delta^2, \tag{4.8}$$

which is valid for any value of the irrational parameter ω.

2. Thue-Morse sequence

The binary Thue-Morse sequence [28] is generated by the substitutions $0 \to 01$ and $1 \to 10$, leading to $0110100110010110\ldots$. The wandering exponent is $\Omega = -\infty$, thus the sequence has bounded fluctuations. Consequently, the diffusion constant is finite and, as shown in Appendix A, can be calculated along the lines of [10] as

$$D_0 = \left(\frac{2}{R^{1/4} + R^{-1/4}}\right)^4. \tag{4.9}$$

The persistence probability is linear for small bias and is given by

$$p_{per}(\delta) = \left(\frac{2}{R^{1/4} + R^{-1/4}}\right)^2 \delta, \tag{4.10}$$

in agreement with the result of Ref. [13], when properly translated. Since the Thue-Morse sequence is reflection symmetric, the persistence probability is the same at both ends of the system, $p_{per}(\delta) = \overline{p}_{per}(\delta)$, so that the relation in Eq. (4.8) is again fulfilled.

Analyzing the results obtained for environments with bounded fluctuations we are led to the following conclusions.

(i) Predictions of the relevance-irrelevance criterion are fully satisfied: The diffusion constant is finite and the critical exponents $(\tau = \psi = \chi = 1)$ take the same values as in homogeneous environments.

(ii) In both examples, the diffusion constant D_0 is invariant under the transformation $R \to 1/R$. This is in agreement with the fact that $D_0(R)$ is maximal for the homogeneous system, i.e., at $R=1$.

(iii) For symmetric sequences the persistence probabilities p_{per} and $\overline{p}_{per}$ are equal, whereas for asymmetric sequences they satisfy the relation $p_{per}(R) = \overline{p}_{per}(1/R)$.

B. Aperiodic environments with unbounded fluctuations

The aperiodic environments display unbounded fluctuations when the wandering exponent $\Omega > 0$. Consequently, the diffusion constant in Eq. (2.5) vanishes in the infinite periodic approximant limit and the persistence probability in Eq. (2.7), like the surface magnetization in the TIM, has an anomalous behavior.

To obtain a qualitative picture for the behavior of the RW in such an environment, we estimate the leading eigenvalue of the FP operator λ_{min} in a finite system of size L, taking into account that the relevant time scale of the process is given by $t \sim \lambda_{min}^{-1}$.

According to Eq. (3.12), the size dependence of λ_{min} for a sequence with unbounded fluctuations is primarily determined by the product $\Pi_{i=1}^{L-1} \epsilon_i^{-1} \sim \exp(-\text{const} \times L^{\Omega})$. Thus, at criticality, the leading eigenvalue has a stretched exponential size dependence

$$\lambda_{min} \sim \exp(-\text{const} \times L^{\Omega}). \tag{4.11}$$

This implies an ultraslow diffusion process, which takes place on a logarithmic time scale

$$\langle X^2(t) \rangle \sim (\ln t)^{2/\Omega}. \tag{4.12}$$

In a random environment, with $\Omega = 1/2$, Eq. (4.12) corresponds to the Sinai diffusion in Eq. (1.1). Next we consider the persistence probability p_{per} and analyze its properties for an exactly solvable environment with unbounded fluctuations.

Rudin-Shapiro environment: Persistence probability

The Rudin-Shapiro sequence [28] is obtained via substitutions on pairs of digits 00→0001, 01→0010, 10→1101, and 11→1110 such that starting on 00, one generates the sequence 0001001000011101 . . . . The wandering exponent of the sequence $\Omega = 1/2$ is the same as for a random environment.

In the critical situation, the persistence probability can be exactly calculated using the methods of [14] and the correspondence in Eq. (3.8). One obtains completely different behaviors for $R > 1$ and $R < 1$: (i) For $R > 1$, the persistence probability remains finite for an infinite system at criticality,

$$p_{per} = 1 - \frac{R^{1/2}}{1 - R^{1/2} + R}, \tag{4.13}$$

thus the transition in the persistence properties is *discontinuous* as the bias δ changes sign; (ii) for $R < 1$, the persistence probability has an anomalous size dependence at the critical point,

$$p_{per} \sim \exp(-\text{const} \times L^{1/2}), \tag{4.14}$$

which goes to zero in the infinite-size limit.

Reading the sequence from the other end amounts to exchanging the digits 1 and 0, which is equivalent to the transformation $R \to 1/R$. Thus the persistence probability of the critical Rudin-Shapiro environment has very different properties for the two ends of the chain: While it stays finite at one end, it vanishes at the other. At this point one may ask how the persistence probability behaves if the system starts at an arbitrary position i along the sequence. One can answer by using the correspondences in Sec. III and the known results for the surface magnetization of the Rudin-Shapiro TIM [29]. *Typically*, i.e., with probability one, the critical point persistence vanishes, as in Eq. (4.14). However, there is a fraction of starting points, the so-called *rare events* $p_{rare} \sim L^{-\Theta_{av}}$, for systems of size L, where the persistence probability is finite $p_{per} = O(1)$, as in Eq. (4.13). Then the *average persistence*, which is obtained by averaging over the starting positions, scales as the probability of the rare events, thus

$$[p_{per}(L)]_{av} \sim p_{rare}(L) \sim L^{-\Theta_{av}}, \tag{4.15}$$

where $[\ ]_{av}$ denotes an average over the starting position. The same scenario remains valid for other aperiodic sequences with unbounded fluctuations. Translating exact results for the TIM [29] into the RW language, the persistence exponent Θ_{av} is connected to the wandering exponent Ω of the sequence via

$$\Theta_{av} = 1 - \Omega. \tag{4.16}$$

To summarize, the diffusion process in an aperiodic environment with unbounded fluctuations is anomalous: (i) The average displacement grows logarithmically in time and (ii) the persistence probability is *not self-averaging*; its averaged dependence involves the fluctuation exponent Ω of the environment.

C. Aperiodic environments with marginal fluctuations

Aperiodic environments with marginal fluctuations are characterized by a wandering exponent $\Omega = 0$. The critical behavior of the TIM with marginally aperiodic couplings is nonuniversal and several critical exponents vary continuously with the strength of the aperiodic perturbation. From the correspondences presented in Sec. III, it follows that the diffusive behavior of the RW in this type of environment is also anomalous. Generally, the diffusion constant D_0 vanishes and both the diffusion exponent $\psi = \psi(R) < 1$ and the persistence exponent $\chi = \chi(R) > 1$ are continuous functions of the asymmetry parameter R.

For aperiodic environments with marginal fluctuations, the scaling behavior of the eigenvalues of the FP operator at the top of the spectrum can be obtained exactly by a renormalization group (RG) transformation, as introduced for the TIM in Ref. [15], later applied to different sequences in [16] and generalized in [17].

The essence of the method is a decimation procedure such that, after one step, a fraction $1/b$ of the original lattice sites are left. The master equation for the renormalized system can be cast into the same form as the original one, provided a finite set of appropriate new parameters are introduced into the original equation (parameter space). The linearized transformation at the fixed point corresponding to $\lambda^*=0$ gives the gap exponent y_λ as indicated in Eq. (2.16). To construct explicitly the RG equations, we refer to the related work on the TIM in Refs. [15–17]. The same type of RG procedure can be used to calculate the persistence exponent, but one can also simply deduce it from a finite-size-scaling analysis of Eq. (2.7) at criticality, as shown in Appendix B for a specific sequence. In the following we present results translated from Refs. [15,16] for several environments with marginal fluctuations.

1. Period-doubling environment

Using the substitutions $0\to01$ and $1\to00$ and starting with 0, one generates the period-doubling sequence [30] 0100010101000100 . . . , which, apart from the last digit, is symmetric and has a vanishing wandering exponent. The diffusion exponent in the period-doubling environment is given by

$$\psi=\frac{\ln 2}{\ln(R^{1/6}+R^{-1/6})}, \tag{4.17}$$

whereas the persistence exponents, which are the same at both ends due to symmetry, read

$$\chi=\overline{\chi}=\frac{\ln(R^{1/6}+R^{-1/6})}{\ln 2}, \tag{4.18}$$

as shown in Appendix B.

2. Paper-folding environment

The paper-folding sequence can be generated by a recurrent folding of a sheet of paper [28]. It corresponds to the two-letter substitutions $00\to0010$, $11\to0111$, $10\to0110$, and $01\to0011$. Starting with 00 one arrives at 0010011000110110 . . . . This environment has the same ($R\leftrightarrow1/R$) type of asymmetry as for the Rudin-Shapiro sequence, if one forgets the last, irrelevant, digit. The wandering exponent vanishes.

The diffusion exponent in the paper-folding environment is

$$\psi=\frac{\ln 2}{\ln(R^{1/4}+R^{-1/4})}. \tag{4.19}$$

The persistence exponents are different at the two ends of the system and are given by

$$\chi=\frac{\ln(1+R^{-1/2})}{\ln 2},\quad \overline{\chi}=\frac{\ln(1+R^{1/2})}{\ln 2}. \tag{4.20}$$

3. Hierarchical environment

Here we consider the Huberman-Kerszberg hierarchical environment [31], where the positions i of the digits f_i satisfy the relation

$$i=2^{f_i}(2l+1),\quad l=0,1,2,\ldots, \tag{4.21}$$

thus the sequence starts as 0102010301020104 . . . . The diffusion problem in the same environment with symmetric transition rates has already been studied before [6].

In the nonsymmetric case the diffusion exponent is

$$\psi=\frac{\ln 2}{\ln(R^{1/2}+R^{-1/2})}. \tag{4.22}$$

The persistence exponents are different at the two ends of the system and are given by

$$\chi=\frac{\ln(1+R^{-1})}{\ln 2},\quad \overline{\chi}=\frac{\ln(1+R)}{\ln 2}. \tag{4.23}$$

One can also easily solve the diffusion problem for generalized hierarchical environments following the solution of the corresponding TIM in Ref. [16].

To summarize, the diffusion in marginally aperiodic environments is anomalous. Both the diffusion exponent ψ and the persistence exponents $\chi,\overline{\chi}$ are continuous functions of the parameter R. However, they satisfy the scaling relation

$$\chi(R)+\overline{\chi}(R)=\frac{2}{\psi(R)}, \tag{4.24}$$

which is a consequence of Eq. (3.12).

V. RELEVANCE-IRRELEVANCE CRITERION IN HIGHER DIMENSIONS

The relevance-irrelevance criterion of Sec. II can be generalized for a d-dimensional environment, where the nonsymmetric transition rates $w_{\mathbf{r},\mathbf{r}'}$ are perfectly correlated in $d-D$ dimensions. Thus they vary in $D\leqslant d$ dimensions and the fluctuations are characterized by a wandering exponent Ω.

In our considerations the basic role is played by the displacement probability $P(L)$, which measures the fraction of walks that have moved to a distance L from their starting position during time $t\sim L^2$, which is the characteristic time scale for a homogeneous medium. In a one-dimensional homogeneous environment with a weak uniform bias $0<\delta_u=\ln(w_\leftarrow/w_\rightarrow)\ll L^{-1}$, the displacement probability in the unfavorable direction is $P_u(L)\sim\exp(-\text{const}\times\delta_u L)$, thus exponentially small.

In the absence of a global bias, an inhomogeneous (random or aperiodic) environment does not generally favor a net displacement of the particle. However, due to the fluctuations in the transition rates, the motion can be effectively biased locally. We now estimate the effective force or average local bias $\delta(L)$ inside a domain U_L of linear size L. First, generalizing the expression in Eq. (4.2), we calculate the accumulated value of $\ln(w_{\mathbf{r},\mathbf{r}'}/w_{\mathbf{r}',\mathbf{r}})$ in the domain as

$$\Delta(L)=\sum_{\mathbf{r}\in U_L}\ln(w_{\mathbf{r},\mathbf{r}'}/w_{\mathbf{r}',\mathbf{r}})\sim\ln RL^{D\Omega}. \tag{5.1}$$

Then the averaged local bias along the inhomogeneous directions is given by

$$\delta(L)\sim\frac{\Delta(L)}{L^D}\sim L^{-D(1-\Omega)}. \tag{5.2}$$

Thus the displacement probability, in analogy with the uniform case, is given by

$$P(L)\sim\exp[-\text{const}\times\delta(L)L]\sim\exp[-\text{const}\times L^{1-D(1-\Omega)}]. \tag{5.3}$$

Now, depending on the sign of the exponent

$$\phi=1-D(1-\Omega), \tag{5.4}$$

the behavior of the displacement probability and, consequently, the diffusion properties are different.

(i) For $\phi<0$, i.e., for $\Omega<1-1/D$, the displacement probability has no exponential size dependence, thus one has the same type of diffusive behavior as for homogeneous systems with zero bias. Consequently, the environment does not modify the normal diffusive motion of the particle and therefore this type of perturbation is *irrelevant*.

(ii) On the other hand, for $\phi>0$, i.e., for $\Omega>1-1/D$, the displacement probability decays exponentially with L^{ϕ}, thus this type of environment is *relevant* for the diffusive properties. The relation between the relevant time scale t, which is proportional to the characteristic number of steps needed to have a displacement L, and the length scale L is obtained as

$$t\sim P(L)^{-1}\sim\exp(\text{const}\times L^{-D(1-\Omega)+1}). \tag{5.5}$$

Thus the mean square displacement, which is proportional to L^2, grows on a logarithmic scale as

$$\langle X^2(t)\rangle\sim(\ln t)^{2/[1-D(1-\Omega)]}. \tag{5.6}$$

(iii) The borderline case $\phi=0$ corresponds to the marginal situation where the L dependence in the exponential of Eq. (5.3) can be logarithmic, leading to nonuniversal diffusive behavior, as observed in Sec. IV for one-dimensional systems.

We note that the above relevance-irrelevance criterion is in complete agreement with the exact results we obtained above for $d=D=1$. For example, Eqs. (4.12) and (5.6) coincide in this case. On the other hand, in the case of uniform disorder with $d=D$ and $\Omega=1/2$, the borderline dimension predicted by Eq. (5.4) is $d^*=2$, which is in agreement with the results of RG investigations [32].

VI. DISCUSSION

In this paper we have studied the scaling properties of the Brownian motion in inhomogeneous environments where the transition rates are asymmetric and their variation follows some quasiperiodic, aperiodic, or hierarchical rules. It has been shown that the diffusive motion of a particle in such environments can be anomalous and a relevance-irrelevance criterion has been formulated, which allows one to predict the different scenarios. In one dimension, we have obtained many exact results that all are in agreement with the abovementioned relevance-irrelevance criterion. In these one-dimensional calculations, we have made use of an exact mathematical correspondence between the master equation of the RW and the eigenvalue problem for the energy of the free-fermionic excitations of an inhomogeneous TIM. This correspondence has been exploited to obtain relations between different physical quantities in the two problems. The analytical results previously obtained for the TIM have been translated into exact results for the diffusion problem.

At this point we note that there is another model of statistical physics, the *directed walk*, which is also closely connected to the TIM. As was shown in Ref. [15], the scaling properties of the directed walk are connected to the eigenvalues of Eq. (3.6) at the *top of the spectrum*. Thus the three problems (Ising model, random walk, and directed walk) are inherently related; the complete solution of any of those contains the necessary information about the properties of the two others. In particular, one single RG transformation describes the scaling properties of the three models: The fixed point at $\Lambda=0$ for the TIM governs the critical properties of the Ising model and that of the Brownian motion, whereas the fixed point at the top of the spectrum is connected to the properties of the directed walk.

Next one can show that the parametrization of the transition probabilities, $w_{i+1,i}=1$ below Eq. (2.2), does not affect the conclusions of the paper. The relevance-irrelevance criterion in Sec. II is evidently unaffected by this restriction and in the marginal situation the nonuniversal exponents $\chi,\overline{\chi}$, and ψ are also insensitive to this parametrization. For the persistence exponents it follows from the fact that in Eqs. (2.7) and (3.13) only the ratio of the transition rates appears. From Eq. (4.24) or, more generally from the relation (3.12), the same conclusion is reached for the exponent ψ. This result can be obtained also by analyzing the structure of the systematic RG technique of Ref. [17].

The properties of the Brownian motion in a relevant aperiodic environment are in many respects similar to those in disordered media. In the critical situation, i.e., in the absence of global bias, the diffusion is ultraslow in both cases and the corresponding relations (1.1) and (4.12) are analogous. In the off-critical situation, however, there is an important difference between the diffusive properties in the two environments. In a disordered media for small enough global bias, such as $0<\delta<\delta_+$, the diffusive motion of the particle is anomalous and the average displacement of the walker grows as a power law $[\langle X(t)\rangle]_{av}\sim t^{\mu}$, with a δ-dependent exponent $0<\mu(\delta)<1$ [33]. This regime is in complete correspondence with the Griffiths-McCoy phase of the random transverse-field Ising spin chain [21]. As shown very recently in Ref. [34], this anomalous diffusion regime is absent for relevant aperiodic environments.

ACKNOWLEDGMENTS

We thank Dragi Karevski for useful discussions. This work was supported by the French-Hungarian cooperation program ''Balaton'' (Ministère des Affaires Étrangères–OMFB), the Hungarian National Research Fund under Grants Nos. OTKA TO23642, OTKA TO25139, and OTKA TO15786, and the Ministery of Education under Grant No. FKFP 0765/1997. H. R. was supported by the Deutsche Forschungsgemeinschaft (DFG). The Laboratoire de Physique des Matériaux is Unité Mixte de Recherche CNRS No. 7556.

APPENDIX A: DIFFUSION CONSTANT AND PERSISTENCE PROBABILITY FOR THE FIBONACCI AND THUE-MORSE ENVIRONMENTS

Using Eq. (4.1), the expression for the diffusion constant D_0 in Eq. (2.5) can be rewritten at criticality as

$$\frac{1}{D_0}=\frac{1}{N^2}\sum_{n=1}^{N}\sum_{i=1}^{N}\epsilon_c^{-n-i}\prod_{j=1}^{i}R^{-f_{j+n}}=\frac{1}{N^2}\sum_{n=1}^{N}\sum_{i=1}^{N}R^{g_n-g_{n+i}}, \tag{A1}$$

where $g_i=n_i-i\rho_\infty$. Similarly, the persistence probability in Eq. (2.7) is given at criticality by

$$\frac{1}{p_{per}(L)}=1+\sum_{i=1}^{L}R^{-g_i}. \tag{A2}$$

For the aperiodic environments, in the limit $N\to\infty$ and $L\gg 1$, respectively, the powers of R in Eqs. (A1) and (A2) can be replaced by their averaged values.

(i) For the Fibonacci sequence, g_i+1/ω is the fractional part of $(i+1)/\omega$, which is uniformly distributed over $[0,1]$ for ω irrational. Thus, in the limit $N\to\infty$, the average in Eq. (A1) can be replaced by an integral

$$\frac{1}{D_0}=\int_0^1 R^g dg\int_0^1 R^{-g}dg, \tag{A3}$$

which leads to the result given in Eq. (4.4). In the same way, for the persitence probability, Eq. (A2) can be rewritten as

$$\frac{1}{p_{per}}=LR^{1/\omega}\int_0^1 R^{-g}dg, \tag{A4}$$

which is evaluated in Eq. (4.5) using the correspondence between δ and L^{-1}.

(ii) For the Thue-Morse sequence, the average can be simply performed by noticing that $f_{2i-1}+f_{2i}=1$, 01 and 10 appearing with the same probability, thus $g_{2i}=0$ with probability 1/2 and $g_{2i+1}=\pm 1/2$, each with probability 1/4. Then considering the different parity combinations for n and i in Eq. (A1), one obtains the expression (4.9). In the same way, Eq. (A2) leads to Eq. (4.10) for the persistence probability.

APPENDIX B: FINITE-SIZE SCALING CALCULATION OF THE PERSISTENCE EXPONENT FOR THE PERIOD-DOUBLING ENVIRONMENT

For the period-doubling sequence, $f_{2i}=1-f_i$ and $f_{2i+1}=0$, so that

$$n_{2i}=n_{2i+1}=\sum_{k=1}^{i}f_{2k}=i-n_i. \tag{B1}$$

On a system with size $4L$, one has

$$\frac{1}{p_{per}(4L)}=S_{4L}(\epsilon,R)=1+\sum_{i=1}^{4L}\epsilon^{-i}R^{-n_i} \tag{B2}$$

and, splitting the sum into odd and even parts,

$$S_{4L}(\epsilon,R)=1+\epsilon^{-1}+\sum_{i=1}^{2L}\epsilon^{-2i}R^{-n_{2i}}+\sum_{i=1}^{2L-1}\epsilon^{-2i-1}R^{-n_{2i+1}}=1+\epsilon^{-1}+\sum_{i=1}^{2L}(\epsilon^2R)^{-i}R^{n_i}+\epsilon^{-1}\sum_{i=1}^{2L-1}(\epsilon^2R)^{-i}R^{n_i}\simeq(1+\epsilon^{-1})S_{2L}(\epsilon^2R,R^{-1}). \tag{B3}$$

Iterating one step further, the last relation in Eq. (B3) leads to

$$S_{4L}(\epsilon,R)=(1+\epsilon^{-1})(1+\epsilon^{-2}R^{-1})S_L(\epsilon^4R,R). \tag{B4}$$

At the critical point $\epsilon_c=R^{-1/3}$ this gives

$$S_{4L}(\epsilon_c,R)=(R^{1/6}+R^{-1/6})^2S_L(\epsilon_c,R). \tag{B5}$$

Since $p_{per}\sim\delta^\chi\sim L^{-\chi}$, one immediately recovers the persistence exponent given in Eq. (4.18).

[1] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981).
[2] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[3] Ja. G. Sinai, Theor. Probab. Appl. **27**, 247 (1982).
[4] P. G. de Gennes, La Recherche **7**, 919 (1976).
[5] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).
[6] A. Giacometti, A. Maritan, and A. L. Stella, Int. J. Mod. Phys. B **5**, 709 (1991).
[7] H. Rieger, Physica A **224**, 267 (1996); J. P. Bouchaud, L. F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin Glasses and Random Fields*, edited by A. P. Young (World Scientific, Singapore, 1998); L. Laloux and P. Le Doussal, Phys. Rev. E **57**, 6296 (1998).
[8] G. Langie and F. Iglói, J. Phys. A **25**, L487 (1992).
[9] C. F. Majkrzak, J. Kwo, M. Hong, Y. Yafet, D. Gibbs, C. L. Chien, and J. Bohr, Adv. Phys. **40**, 99 (1991).
[10] J. M. Luck, J. Stat. Phys. **72**, 417 (1993).
[11] F. Iglói, J. Phys. A **26**, L703 (1993); J. M. Luck, Europhys. Lett. **24**, 359 (1993).
[12] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[13] L. Turban, F. Iglói, and B. Berche, Phys. Rev. B **49**, 12 695 (1994).
[14] F. Iglói and L. Turban, Europhys. Lett. **27**, 91 (1994).
[15] F. Iglói and L. Turban, Phys. Rev. Lett. **77**, 1206 (1996).
[16] F. Iglói, L. Turban, D. Karevski, and F. Szalma, Phys. Rev. B **56**, 11 031 (1997).
[17] J. Hermisson, U. Grimm, and M. Baake, J. Phys. A **30**, 7315 (1997).
[18] F. Iglói and P. Lajkó, J. Phys. A **29**, 4803 (1996).
[19] P. E. Berche, C. Chatelain, and B. Berche, Phys. Rev. Lett. **80**, 297 (1998).
[20] B. Derrida, J. Stat. Phys. **31**, 433 (1983).

[21] F. Iglói and H. Rieger, Phys. Rev. E **58**, 4238 (1998).
[22] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961); S. Katsura, Phys. Rev. **127**, 1508 (1962); P. Pfeuty, Ann. Phys. (Paris) **57**, 79 (1970).
[23] F. Iglói and H. Rieger, Phys. Rev. B **57**, 11 404 (1998).
[24] I. Peschel, Phys. Rev. B **30**, 6783 (1984).
[25] M. Queffélec, *Substitutional Dynamical Systems–Spectral Analysis*, edited by A. Dold and B. Eckmann, Lecture Notes in Mathematics Vol. 1294 (Springer, Berlin, 1987).
[26] The substitution rule used here to generate the Fibonacci sequence corresponds to the product of two substitutions with the more usual rule $0\to01$, $1\to0$. It has the advantage of leading by iteration, starting with 0, to a succession of finite sequences that are stable from both ends: 0, 010, 01001010, ...
[27] L. Turban and B. Berche, Z. Phys. B **92**, 307 (1993).
[28] M. Dekking, M. Mendès-France, and A. van der Poorten, Math. Intelligencer **4**, 130 (1983).
[29] F. Iglói, D. Karevski, and H. Rieger, Eur. Phys. J. B **1**, 513 (1998).
[30] P. Collet and J. P. Eckmann, *Iterated Maps in the Interval as Dynamical Systems* (Birkhäuser, Boston, 1980).
[31] H. A. Simon and A. Ando, Econometrica **29**, 111 (1961); B. A. Huberman and M. Kerszberg, J. Phys. A **18**, L331 (1985).
[32] J. M. Luck, Nucl. Phys. B **225**, 169 (1983); D. S. Fisher, Phys. Rev. A **30**, 960 (1984).
[33] B. Derrida and Y. Pomeau, Phys. Rev. Lett. **48**, 627 (1982).
[34] F. Iglói, D. Karevski, and H. Rieger, Eur. Phys. J. B **5**, 613 (1998).