Constrained spin-dynamics description of random walks on hierarchical scale-free networks

Jae Dong Noh
Department of Physics, Chungnam National University, Daejeon 305-764, Korea

Heiko Rieger
Theoretische Physik, Universität des Saarlandes, 66041 Saarbrücken, Germany

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We study a random walk problem on the hierarchical network which is a scale-free network grown deterministically. The random walk problem is mapped onto a dynamical Ising spin chain system in one dimension with a nonlocal spin update rule, which allows an analytic approach. We show analytically that the characteristic relaxation time scale grows algebraically with the total number of nodes N as $T \sim N^z$. From a scaling argument, we also show the power-law decay of the autocorrelation function $C_{\boldsymbol{\sigma}}(t) \sim t^{-\alpha}$, which is the probability to find the Ising spins in the initial state $\boldsymbol{\sigma}$ after t time steps, with the state-dependent nonuniversal exponent α. It turns out that the power-law scaling behavior has its origin in a quasiultrametric structure of the configuration space.

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I. INTRODUCTION

Complex networks, as, for instance, represented by the Internet, the social acquaintance network between individuals, biological networks of interacting proteins, and others (see Ref. [1] for further examples), became recently a central research focus in statistical physics. In general a network consists of a set of nodes (sites or vertices) and a set of edges (bonds or arcs), connecting the nodes with one another. A system with many interacting degrees of freedom, e.g., computers, individuals, proteins, etc., or generally called agents, can be modeled by a network by identifying the agents as the nodes and the interaction between them as the edges. Real world networks have neither a regular structure (such as, for instance, periodic lattices or grid graphs have) nor a fully random structure [2]. They rather display a broad distribution of the degree, where the degree K of a node is the number of neighbors connected to it. Some networks, the so-called scale-free networks [3], display a power-law degree distribution $P(K) \sim K^{-\gamma}$, which is found in various disciplines.

The heterogeneous structure of scale-free networks has a significant influence on thermodynamic or dynamic systems embedded into them. For instance, the natures of equilibrium [4] or nonequilibrium [5] phase transitions are quite different from those observed in corresponding systems on regular periodic lattices. In the present work we are interested in the nature of diffusive and relaxational dynamics performed by a random walker in scale-free hierarchical network [6]. As a very recent application we note that in the context of peer-to-peer computer networks random walk search strategies have been proposed [7–9], in which a query message is forwarded to a randomly chosen neighbor at each step until the desired object (typically a particular data set) is found. In view of these algorithmic developments it appears therefore quite natural and important to study random walks on complex networks. In addition, the random walk is a fundamental stochastic process [10] and turns out to be a useful tool in characterizing the structure of complex networks [11–13].

In regular networks of periodic lattices in D dimension, the random walk motion is characterized by normal diffusion which is characterized by a length scale that grows algebraically as $\xi \sim t^{1/2}$ in time t. The exponent 1/2 is universal, i.e., it does not depend on the microscopic details of the lattice—the only condition being that only nearest neighbor jumps on a regular D-dimensional lattice are allowed. The autocorrelation function $C(t)$ or the return probability to the initial node in t time steps decays algebraically as $C(t) \sim t^{-D/2}$. On random networks, on the other hand, the autocorrelation function shows a stretched-exponential decay as $C(t) \sim e^{-at^{\beta}}$ with $\beta = 1/3$ [14].

Random walks were also studied in the small-world network of Watts and Strogatz [2], which interpolates between regular networks and random networks by stochastically changing connections between nodes with a particular rewiring probability p_W. In essence a small-world network is obtained from a regular network with edges of fraction p_W being replaced by shortcuts connecting pairs of nodes selected randomly. For nonzero p_W, an interesting crossover behavior is observed [15,16]: A random walk obeys the scaling law for regular networks for short times $t \ll \tau$, and then that for the random networks for large times $t \gg \tau$. The crossover time scale τ is determined by the time interval at which a random walker hits shortcuts. Since the mean distance between shortcuts is $\xi \sim p_W^{-1}$, the crossover time scales as $\tau \sim \xi^2 \sim p_W^{-2}$. For $t \gg \tau$, it is numerically found that the autocorrelation function also shows a stretched-exponential decay as $C(t) \sim e^{-at^{\beta}}$ with $\beta \simeq 1/3$ [17,18].

There has been a growing interest recently in the study of random walks on scale-free networks [19,20]. In this paper, we focus on random walks on a hierarchical network, which is a model for a scale-free network with a modular structure [6]. Unlike most scale-free network models it is a deterministic network as those of Jung *et al.* [21] and Dorogovtsev *et al.* [22]. Due to its deterministic nature a number of characteristic structural features are known exactly [23]. As we will see in the following, we can study various properties of the random walk analytically. The analytic results will shed light on the stochastic processes in general scale-free networks.

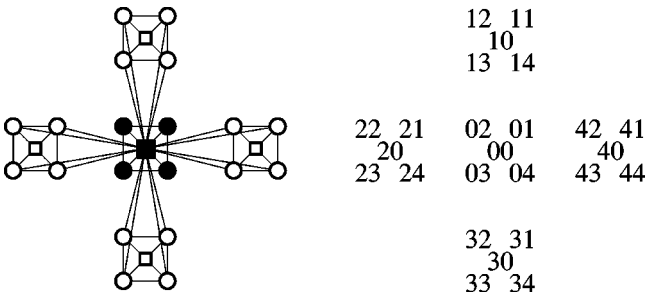

FIG. 1. The configuration and the coordinate representation of $\mathcal{G}_2$ with $M=5$. The hub is represented with a filled square, and the peripheral nodes are with empty circles.

The paper is organized as follows: In Sec. II, the hierarchical network model and the random walk is introduced. Our results for the scaling laws for the relaxation time and the autocorrelation functions are presented in Sec. III. These results are derived with the help of an exact mapping of the random walk problem to a constrained dynamics of an Ising spin chain; the details of the mapping are described in Sec. IV. We also find that a random walk on a hierarchical network is similar to the diffusion in ultrametric space, which is elaborated in Sec. V. Finally we summarize our work in Sec. VI.

II. MODEL

Some biological networks which are scale-free exhibit a modular structure, which is not incorporated into most scale-free network models. The hierarchical network has been proposed as a model for the scale-free networks with the modular structure [6]. It is constructed iteratively starting from a seed (first generation) $\mathcal{G}_1$ consisting of a *hub* and $(M-1)$ *peripheral nodes*. They are fully connected with each other. It is useful to represent the hub and the peripheral nodes with the coordinates (0) and (y), where y is an integer $1 \leqslant y < M$ [23]. Nodes in $\mathcal{G}_G$, the network of the Gth generation, are identified via coordinates that are G tuples of integers $(\boldsymbol{x})=(x_G, \ldots, x_1)$.

From a given graph $\mathcal{G}_g$, the next generation network $\mathcal{G}_{g+1}$ is constructed by adding $(M-1)$ copies of $\mathcal{G}_g$ with their peripheral nodes connected to the hub of the original $\mathcal{G}_g$. The original hub and the peripheral nodes in the copies become the hub and peripheral nodes of $\mathcal{G}_{g+1}$, respectively. Then, each node whose coordinate was $(\boldsymbol{x})$ is assigned to $(0\boldsymbol{x})$ if it belongs to the original $\mathcal{G}_g$ or to $(y\boldsymbol{x})$ with $1 \leqslant y < M$ if it belongs to the yth copy of $\mathcal{G}_g$. So the hub has $x_n = 0$ for all n and a peripheral node has $x_n \neq 0$ for all n. There are M^G nodes in $\mathcal{G}_G$, $(M-1)^G$ of which are peripheral nodes. Figure 1 shows the configuration and the coordinate representation of $\mathcal{G}_2$ with $M=5$. The iteration can be repeated indefinitely and the emerging network is scale-free for $M \geqslant 3$ with the degree distribution exponent $\gamma = 1 + \ln M/\ln(M-1)$ [23].

The node connectivity is represented by the adjacency matrix A_{ji}; $A_{ji}=1$ if a node i is connected to j or 0 otherwise. The network is undirected, hence $A_{ij}=A_{ji}$ and the connectivity is easily described in terms of the coordinates [23]. Hereafter, we will use x for a dummy index from 0 to $M-1$, while y from 1 to $M-1$, and we denote the m tuple of 0 as $\mathbf{0}_m$.

The network growth rule implies the following.

(a) The existence of connections of mth generation hub to all mth generation peripheral nodes, more precisely in coordinate language, nodes $(\boldsymbol{x})$ with $x_i=0$ for $i=1, \ldots, m$ and $x_{m+1}=y_{m+1}\neq 0$ are connected to the following nodes:

$$(\cdots y_{m+1}\mathbf{0}_m) \leftrightarrow (\cdots y_{m+1}\mathbf{0}_{m-n}y_n \cdots y_1) \tag{1}$$

with $1 \leqslant n \leqslant m$.

(b) It implies the existence of connections between peripheral nodes and lower level hubs plus connections to other peripheral nodes within the same elementary unit; in coordinate language, a node $(\boldsymbol{x})$ with $x_i=y_i\neq 0$ for $i=1, \ldots, m$ and $x_{m+1}=0$ is connected to the following nodes:

$$(\cdots 0y_m \cdots y_1) \leftrightarrow \begin{cases} (\cdots 0y_m \cdots y_2 y_1') \\ (\cdots 0y_m \cdots y_{n+1}\mathbf{0}_n) \end{cases} \tag{2}$$

with $y_1' \neq y_1$ and $1 \leqslant n \leqslant m$.

We study a discrete time random walk on the network. This stochastic process is defined by the following rules: The walker at node i and time t selects one of the neighbors of i to which i is connected and jumps to this neighbor at time $t+1$. Thus the transition probability for a jump from a node i to a node j is given by $\omega_{ji}=A_{ji}/K_i$, where A_{ji} is the adjacency matrix and $K_i=\Sigma_j A_{ji}$ is the degree of the node i.

This stochastic process in discrete time is described by a master equation for the time evolution of $P_i(t)$, the probability finding the walker at node i and time t. The master equation reads $P_i(t+1)=\Sigma_j \omega_{ij} P_j(t)$. Equivalently, defining the state vector $|P(t)\rangle \equiv \Sigma_i P_i(t)|i\rangle$ with $|i\rangle$ being the state in which the walker is at node i, one can rewrite the master equation as $|P(t+1)\rangle = \hat{\boldsymbol{U}}|P(t)\rangle$, where $\hat{\boldsymbol{U}}$ is the transition operator whose elements are $(\hat{\boldsymbol{U}})_{ji}=\omega_{ji}$.

In the infinite time limit $t \to \infty$ the probability distribution converges to the stationary state distribution P_i^∞, which is given by $P_i^\infty = K_i/\mathcal{N}$ with $\mathcal{N} \equiv \Sigma_i K_i$ for the random walk on arbitrary undirected network [12]. In the hierarchical network the degrees of all nodes are known exactly [23]. For instance, the hub has the largest degree

$$K_h=(M-1)(M-2)^{-1}[(M-1)^G-1] \sim (M-1)^G, \tag{3}$$

and the peripheral node has the degree

$$K_p=(M-2+G). \tag{4}$$

The sum of all degrees is given by

$$\mathcal{N}=(3M-2)(M-1)M^{G-1}-2(M-1)^{G+1} \sim M^G. \tag{5}$$

A quantity of particular interest is the scaling law for the relaxation time T, which is the characteristic time scale for the approach of the probability distribution $P_i(t)$ to the stationary state distribution P_i^∞. Also of interest is the nature of the relaxation dynamics, for which we consider the decay of the autocorrelation function

$$C_S(t) = \langle S|\hat{\boldsymbol{U}}^t|S\rangle, \tag{6}$$

which is the overlap between a state $|S\rangle$ with itself after t time steps. When $|S\rangle = |i\rangle$, it reduces to the returning probability of the random walker to the origin (starting node) i after t time steps. In the limit $t \to \infty$ the autocorrelation function converges to a value determined by the stationary state distribution P^{∞}. The scaling behavior of $C_S(t)$ for $t \ll T$ will be studied for various states $|S\rangle$.

III. RESULTS

In this section we present our main results. They are derived using the exact mapping of our random walk process onto a constrained dynamics of an Ising spin chain. Details of the mapping and the derivations of the formulas deduced from it and used in the present section are delegated to the following section.

A. Relaxation time

Consider the motion of the random walker located initially on a particular node, say (030201). The memory of the initial position will be lost when all components x_i's are flipped at least once, which defines the relaxation time scale T. The node connectivity summarized in Eqs. (1) and (2) tells us that x_i may flip only when all x_j's with $j<i$ are equal to 0 (if $x_i=0$) or all are not equal to 0 (if $x_i \neq 0$). Hence, the random walker should follow the path

$$\begin{aligned}(030201) &\to (030200) \to (0302y_2y_1) \to (030000) \\ &\to (03y_4y_3y_2'y_1') \to (000000) \to (y_6y_5y_4'y_3'y_2''y_1'')\end{aligned}$$

to lose the memory of its initial state.

Each process requires a simultaneous flip of ξ components from zero to nonzero values or vice versa, which may occur after many trials. For instance, the random walker at $(0302y_2y_1)$ may hop to $(0302y_20)$ or (030200) instead of to (030000). When it jumps to a wrong node, say (030200), first it should hop to a node $(0302y_2'y_1')$, and then try another hopping toward the destination. In this respect the dynamics we are considering is of a hierarchical nature. Utilizing this observation we will show in the following section that the associated time scale τ_ξ for the process increases exponentially as $\tau_\xi \sim [M/(M-1)]^\xi$. We define $\kappa \equiv M/(M-1)$ for further use.

Therefore, the relaxation time T, which is given by $T \sim \Sigma_\xi^G \tau_\xi$, scales *exponentially* with G as

$$T \sim \kappa^G. \tag{7}$$

Since $N = M^G$, the relaxation time scales *algebraically* with N as

$$T \sim N^z, \tag{8}$$

with the dynamic exponent

$$z = \ln\kappa / \ln M. \tag{9}$$

B. Autocorrelation function

To be specific we consider the autocorrelation functions for the following states:

(i) H is the state corresponding to the hub

$$|H\rangle = |\mathbf{0}_G\rangle. \tag{10}$$

(ii) P is the state corresponding to the peripheral nodes

$$|P\rangle = \frac{1}{(M-1)^G} \sum_{y_1,\ldots,y_G} |y_G,\ldots,y_1\rangle. \tag{11}$$

(iii) $A1$ and $A2$ are the states

$$|A1\rangle = \frac{1}{(M-1)^{(G/2)}} \sum_{y_2,y_4,\ldots} |\cdots y_4 0 y_3 0\rangle, \tag{12}$$

$$|A2\rangle = \frac{1}{(M-1)^{(G/2)}} \sum_{y_1,y_3,\ldots} |\cdots 0 y_3 0 y_1\rangle \tag{13}$$

with zero and nonzero components alternating.

The stationary state probability distribution is determined by the degree distribution. Since the degree of all nodes is known, it is easy to show that $P_H^{\infty} \sim \kappa^{-G}$, $P_P^{\infty} \sim G\kappa^{-G}$, and $P_{A1}^{\infty} = P_{A2}^{\infty} \sim (M/\sqrt{M-1})^{-G}$ in the large G limit.

The exponential decrease of the stationary state probability and the exponential increase of the relaxation time suggests a power-law decay of the autocorrelation function in time. Indeed, we find that the autocorrelation functions decay algebraically for $t \ll T$ as

$$C_H(t) \sim \frac{1}{G^2}\, t^{-\alpha_H}, \tag{14}$$

$$C_P(t) \sim \frac{1}{G}\, t^{-\alpha_P}, \tag{15}$$

$$C_{A1}(t) \simeq C_{A2}(t) \sim t^{-\alpha_A}, \tag{16}$$

where $\alpha_H = \alpha_P = 1$ and $\alpha_A = \ln(M/\sqrt{M-1})/\ln\kappa > 1$ with $\kappa = M/(M-1)$. Quite remarkably, the decay exponent depends on the state—a manifestation of the fact that the network under consideration is not homogeneous. In addition to the power-law dependency in t, the functions $C_P(t)$ and $C_H(t)$ also decay as $1/G$ and $1/G^2$, respectively, i.e., algebraically with the number of generations in the network. So, the power-law decay in time is observed only in finite systems for the states H and P, since in the limit $G \to \infty$ the functions C_H and C_P vanish.

IV. ISING SPIN CHAIN

In this section, we explain the exact mapping of the random walk problem onto the constrained dynamics of an Ising spin chain.

A. Mapping

Using the coordinate representation of the nodes, one may map the state $|i\rangle$ with a random walker at a node $i=(\boldsymbol{x})$ in $\mathcal{G}_G$ to a spin configuration $|\boldsymbol{x}\rangle = |x_G,\ldots,x_1\rangle$ of an M-state Potts spin chain of length G, where $x_n \in \{0,\ldots,M-1\}$ denotes the state of the spin at site n $(=1,\ldots,G)$ in the chain.

A jump of the walker corresponds to a transition between spin configuration. In this way the connection rules define the time evolution of the spins.

In the context of spin dynamics, it is useful to define a *zero domain* (ZD) and a *nonzero domain* (NZD); the ZD is a domain of spins that are all in the zero state, i.e., $x_i = x_{i+1} = \cdots = x_{i+l} = 0$; and the NZD is one in which all spins are in a nonzero state. In particular, a domain including the spin x_1 will be called a *boundary domain*. The node connectivity imposes the constraint that spins outside the boundary domain cannot flip in a given spin configuration. So it suffices to consider the transition of spins in the boundary domain. Equation (1) implies that spins in a boundary ZD evolve in one time step according to $\hat{\boldsymbol{U}}|\mathbf{0}_m\rangle = \Sigma_{n=1}^{m}(\Sigma_{y_1,\ldots,y_n}|\mathbf{0}_{m-n}y_n \cdots y_1\rangle)/\Omega$ with $\Omega = \Sigma_{n=1}^{m}(M-1)^n$. On the other hand, Eq. (2) implies that spins in a boundary NZD evolve as

$$\hat{\boldsymbol{U}}|y_m \cdots y_1\rangle = (\Sigma_{y_1' \neq y_1}|y_m \cdots y_2 y_1'\rangle + \Sigma_{n=1}^{m}|y_m \cdots y_{n+1}\mathbf{0}_n\rangle)/\Omega'$$

$$\text{with } \Omega' = (M-2+m).$$

Note that the boundary domain size decreases in most cases. It increases only when all spins in the boundary domain flip.

The operator $\hat{\boldsymbol{U}}$ is symmetric under any permutation $y_n \to y_n'$ among nonzero spin states. Taking advantage of the symmetry, we restrict ourselves to the subspace which is invariant under all such permutations. The subspace is spanned by the states $|\boldsymbol{\sigma}\rangle = |\sigma_G \cdots \sigma_1\rangle = |\sigma_G\rangle \otimes \cdots \otimes |\sigma_1\rangle$, where $\sigma_n = \pm$ and

$$|+\rangle \equiv \frac{1}{M-1}\sum_{y=1}^{M-1}|y\rangle, \tag{17}$$

$$|-\rangle \equiv |0\rangle. \tag{18}$$

For example, in $\mathcal{G}_2$ with $M=5$ as shown in Fig. 1, $|--\rangle$ corresponds to the state with the walker at the hub, and the states $|+-\rangle$, $|-+\rangle$, and $|++\rangle$ correspond to the states in which the walker can be found with equal probability on nodes □'s, ●'s, and ○'s, respectively.

One may regard the two-state variable σ as the Ising spin. Then the random walk problem in the subspace reduces to an Ising spin chain with a particular constrained dynamics. In fact, the states defined in the preceding section are equal to the ferromagnetically and antiferromagnetically ordered states:

$$|H\rangle = |--\cdots--\rangle, \tag{19}$$

$$|P\rangle = |++\cdots++\rangle, \tag{20}$$

$$|A1\rangle = |\cdots+-+-\rangle, \tag{21}$$

$$|A2\rangle = |\cdots-+-+\rangle. \tag{22}$$

The Ising spins evolve as follows: As in the Potts spin dynamics, only spins in the boundary domain may flip. A boundary domain $|\pm_m\rangle$ of up/down spins of size m evolves in one time step according to

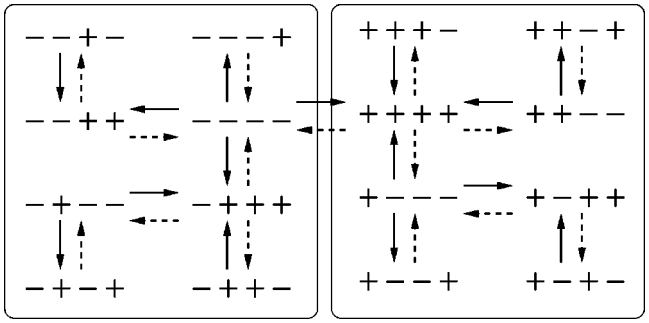

FIG. 2. The configuration space of the Ising spin chain of length $G=4$. Solid (dashed) lines with the arrow represent the transition with the probability $q_{m,n}$ ($p_{m,n}$), where m is the boundary domain size of a source state and n is the number of flipped spins. Self-loops from states with a + boundary domain to themselves with weights $p_{m,0}$ are omitted. The parts inside the boxes with σ_4 being ignored are equivalent to the configuration space of $\mathcal{G}_3$; the configuration space has a hierarchical structure.

$$\hat{\boldsymbol{U}}|+_m\rangle = \sum_{n=0}^{m} p_{m,n}|+_{m-n}-_n\rangle, \tag{23}$$

$$\hat{\boldsymbol{U}}|-_m\rangle = \sum_{n=1}^{m} q_{m,n}|-_{m-n}+_n\rangle, \tag{24}$$

where $p_{m,n} = [(M-2)\delta_{n,0} + (1-\delta_{n,0})]/(M-2+m)$ and $q_{m,n} = (M-1)^n/\Sigma_{k=1}^{m}(M-1)^k$. Each spin state has a different multiplicity factor, so the transition probabilities $p_{m,n}$ and $q_{m,n}$ are not uniform in n. Note that the spin up-down symmetry is broken for $M \geqslant 3$. It is restored for $M=2$, in which case the corresponding network is not scale-free.

In Fig. 2, we illustrate a diagram of the configuration space of the spin chain of length $G=4$ displaying the spin configurations and possible transitions between them. Qualitative features of the Ising spin dynamics are easily read off from the diagram: (i) The configuration space has a tree structure, if one ignores self-loops from the states with a + boundary domain to themselves ($p_{m,0} \neq 0$); (ii) the configuration space has a hierarchical structure, that is, the configuration space of $\mathcal{G}_G$ contains those of $\mathcal{G}_{G'}$ with $G' < G$ as parts (see Fig. 2).

B. Boundary domain growth

The condition that only spins in a boundary domain may flip imposes severe constraint on the spin relaxation dynamics. In a given Ising spin configuration, σ_2 may flip after σ_1 aligns parallel to it, σ_3 may flip after σ_1 and σ_2 align parallel to it, and then, in general, σ_m may flip after all spins σ_n with $n < m$ align parallel to it. In other words, the boundary domain size grows up to m in order to flip σ_m. The boundary domain growth is the essential mechanism in the spin relaxation dynamics. In the language of a domain wall, a domain wall at site $n+1/2$, i.e., $\sigma_{n+1} \neq \sigma_n$, plays a role of a dynamic barrier since it prevents spins σ_m with $m > n$ from flipping.

Consider a spin configuration with a boundary domain of size m. The size of the boundary domain increases only when all m spins inside the domain flip simultaneously. When $n < m$ spins flip, the boundary domain size reduces to n. Then the spin system should grow the boundary domain size up to

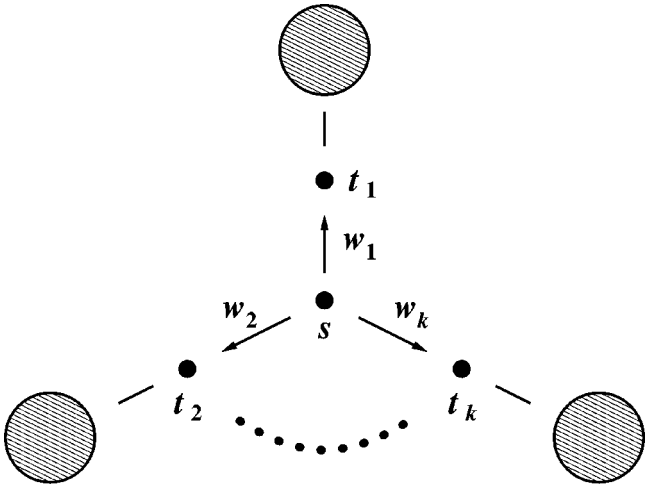

FIG. 3. Treelike network. The shaded areas need not have a tree structure as long as there is no overlap between different parts.

m to return to the initial state and try another flip to increase the boundary domain size. It shows that the boundary growth process has a hierarchical nature, which is inherited from the hierarchical structure of the configuration space.

We investigate the characteristic time scale associated with the boundary domain growth process. Due to the hierarchical nature of the dynamics, we find that the time scale satisfies a recursion relation. To be more specific, we consider the mean first passage time (MFPT) T_m^{+-} (T_m^{-+}), which it takes to flip all spins in the boundary domain of m up (down) spins simultaneously for the first time. Note that such time scales do not depend on spins outside the boundary domain, so they do not depend on the total chain length G.

Before proceeding, we derive a useful formula for the MFPT in a *treelike* structure. Consider a node (or state) s which is connected to k nodes t_i with $i=1,\ldots,k$. The transition probability from s to t_i is given by ω_i, and to itself by ω_0 (see Fig. 3). By the treelike structure, we mean that t_i can be reached from t_j only through s for all pairs of i and j, no matter how many loops there are in the shaded areas. Then, T_i, the MFPT from s to t_i, is given by

$$T_i=\omega_i+\sum_{j\neq i}(2+T_j')\omega_j\omega_i + \sum_{j,j'\neq i}(3+T_j'+T_{j'}')\omega_j\omega_{j'}\omega_i+\cdots,$$

where $T_{j\neq 0}'$ denotes the MFPT from t_j to s and T_0' is set to zero. The first term corresponds to the transition to t_i in a single step, the second term to a round trip via $t_{j\neq i}$ or staying at s followed by the transition to t_i, and so on. The infinite sum can easily be evaluated which yields

$$T_i=\omega_i^{-1}\left(1+\sum_{j\neq i}\omega_j T_j'\right). \tag{25}$$

The configuration space of the Ising spin chain has a tree structure. So we can make use of the formula in Eq. (25). Take a spin state with a boundary domain of m up spins as s in Fig. 3. It is connected to spin states with boundary domains of n down spins $(n=1,\ldots,m-1)$ with the transition probabilities $p_{m,n}$, which leads to

$$T_m^{+-}=p_{m,m}^{-1}\left(1+\sum_{n=1}^{m-1}p_{m,n}T_n^{-+}\right). \tag{26}$$

Likewise, one also obtains that

$$T_m^{-+}=q_{m,m}^{-1}\left(1+\sum_{n=1}^{m-1}q_{m,n}T_n^{+-}\right). \tag{27}$$

After lengthy but straightforward calculations, the recursion relations can be solved exactly to yield

$$T_m^{+-}=(3-2M)+(3M-2)\,\kappa^{m-2}, \tag{28}$$

$$T_m^{-+}=[(3M-2)/M]\,\kappa^{m-1}-1 \tag{29}$$

for $m\geqslant 2$ and $T_1^{+-}=M-1$ and $T_1^{-+}=1$. Recall that $\kappa=M/(M-1)$. The time scales increase *exponentially* with m.

C. Relaxation time

Consider an arbitrary spin configuration $|\boldsymbol{\sigma}\rangle$ with l domain walls at sites $\{m_1+1/2,\ldots,m_l+1/2\}$ with $m_i<m_j$ for $i<j$. The spin state has a boundary domain of size m_1 initially. The spin system loses the memory of the initial state when all spins flip at least once. Note that σ_G is the last spin to flip. So, the characteristic relaxation time is given by the time at which σ_G flips for the first time. It can flip when all spins align ferromagnetically, which requires that spins $\sigma_{n\leqslant m_l}$ align, which also requires that spins $\sigma_{n\leqslant m_{l-1}}$ align, and so on. Therefore the relaxation time is given by $T=\sum_{a=1}^{l}T_{m_a}^{\pm\mp}+T_G^{\pm\mp}$. For example, the relaxation time for a spin state $|+--+\rangle$ is given by $T=T_2^{+-}+T_3^{-+}+T_4^{+-}$.

Since $T_m^{\pm\mp}$ increases exponentially in m, the sum is dominated by the last term $T_G^{\pm\mp}$ for all spin states. Therefore we conclude that the characteristic relaxation time averaged over all states scales as $T\sim T_G^{+-}\sim T_G^{-+}$, which gives $T\sim\kappa^G$, i.e., the important formulas in Eqs. (7) and (8).

D. Autocorrelation

In this section, we derive the scaling laws for the autocorrelation function $C_{\boldsymbol{\sigma}}(t)$. It measures the strength of the memory of the initial state $|\boldsymbol{\sigma}\rangle$ after time t. The spin system loses the memory as more and more spins fluctuate. Due to the hierarchical nature of the spin dynamics, the spin fluctuations grow from one boundary of the chain, namely, from σ_1. So, it is useful to define a length scale $\xi(t)$ which is determined by the condition that $\sigma_n(t)=\sigma_n(0)$ for $n>\xi$ and $\sigma_\xi(t)\neq\sigma_\xi(0)$, where $\boldsymbol{\sigma}(t)$ denotes the spin state at time t. All spins at sites $n\leqslant\xi$ have flipped at least once up to t. For this reason, we will call those sites the *perturbed domain*, and ξ the *perturbed domain size*. Roughly speaking, $\xi(t)$ is the maximum size of the boundary domain up to time t.

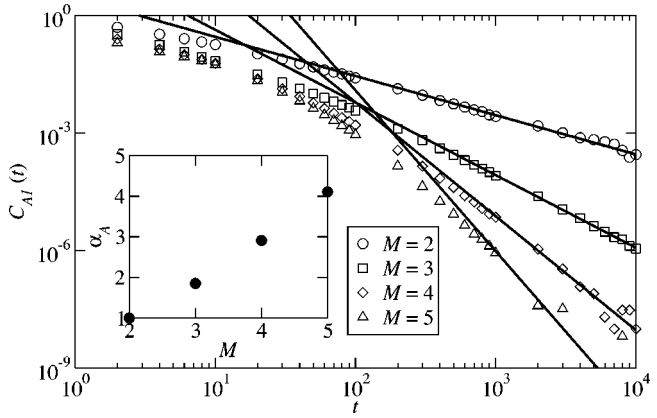

FIG. 4. Numerical results for $C_{A1}(t)$ are represented by the symbol plots for $M=2,3,4,5$ in the Ising spin chain of length $G=100$. The solid lines have the slope given by $\alpha_{A1}(M)=\ln(M/\sqrt{M-1})/\ln\kappa$. The inset shows the plot of α_{A1} vs M.

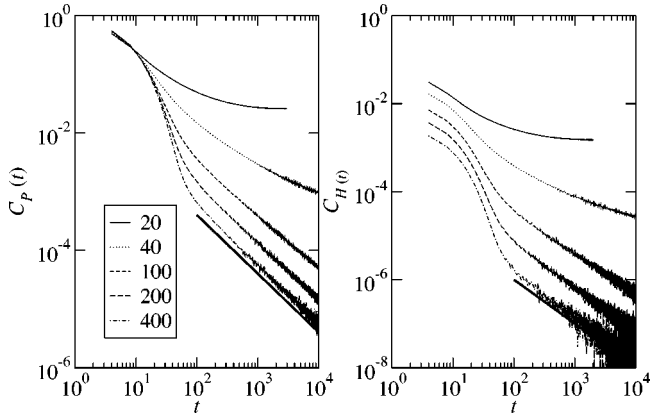

FIG. 5. Numerical results for $C_P(t)$ and $C_H(t)$ for different values of G (as listed in the inset) with $M=5$. The solid lines have the slope -1.

First, consider the antiferromagnetically ordered state $|A1\rangle$ defined in Eqs. (12) and (21). One obtains the same results for the state $|A2\rangle$. It is the linear superposition of $(M-1)^{G/2}$ states, in which the random walker is located at nodes $|\cdots 0y_3 0y_1\rangle$, each of which has the degree $K=M-1$. Hence, its stationary state probability is given by

$$P_{A1}^{\infty}\sim(M-1)^{G/2+1}/\mathcal{N}\sim r_A^{-G} \tag{30}$$

with the chain length G and $r_A=M/\sqrt{M-1}$.

The state $|A1\rangle$ has the highest density of domain walls. In such a state, the perturbed domain grows by removing the domain walls successively. So, the perturbed domain size reaches ξ after the time scale $\tau_\xi\sim\Sigma_{n<\xi}T_n^{\pm\mp}\sim\kappa^\xi$. Note that the time scale t_ξ is of the same order of magnitude as the relaxation time scale of the spin chain of length ξ. It implies that the spins $\sigma_\xi\cdots\sigma_1$ in the perturbed domain are in the stationary state, while those spins outside the perturbed domains are frozen at that time scale. Therefore, $C_A(t_\xi)$ is given by the stationary state probability for the antiferromagnetic state in the chain of length ξ, that is, C_{A1}^{∞} in Eq. (30) with G replaced by ξ to yield $C_{A1}(t_\xi)\sim r_A^{-\xi}$. Eliminating ξ in t_ξ and $C_{A1}(t_\xi)$, we obtain the power-law decay as written in Eq. (16).

We confirmed the analytical results with numerical simulations of the Ising spin chain. Starting from the initial state $|\cdots+-+-\rangle$, a stochastic time evolution is generated using the transition rules in Eqs. (23) and (24) and $C_{A1}(t)$ is measured and averaged over independent runs. In Fig. 4, the numerical results are presented. They are consistent with the analytic results.

For the ferromagnetic states $|P\rangle$, one can apply a similar scaling argument with a little care. It is a linear superposition of $(M-1)^G$ states, in which the random walker is located on peripheral nodes with degree $M-1+G$. So, its stationary state probability is given by

$$P_P^{\infty}=(M-1+G)(M-1)^G/\mathcal{N}\sim G\kappa^{-G}. \tag{31}$$

The state does not contain any domain walls. So in the beginning it evolves quickly creating domain walls into one of states $\{|\eta\rangle\}$ with $\eta=1,\ldots,G$ with the transition probability $p_{G,\eta}\sim 1/G$, where $|\eta\rangle$ denotes a state with the domain wall at $\eta+1/2$, i.e., $\sigma_{n>\eta}=+$ and $\sigma_\eta=-$. After this, the boundary domain growth takes place for each $|\eta\rangle$ independently. After a time scale $t\sim\kappa^\xi$, the spins $\sigma_\xi\cdots\sigma_1$ in the state $|\eta=\xi\rangle$ reach the stationary state with the probability for them to be in the ferromagnetic up state being given by P_P^{∞} in Eq. (31) with G replaced by ξ, i.e., $\xi r_P^{-\xi}$. Therefore the value of the autocorrelation function is given by $C_P(t_\xi)\sim\xi r_P^{-\xi}/G$ where $1/G$ is the transition probability from $|P\rangle$ to $|\eta=\xi\rangle$. Eliminating ξ using $t_\xi\sim\kappa^\xi$, we obtain the result in Eq. (15) in the leading order.

Analogously, the state $|H\rangle$ evolves into one of the states $\{|\zeta\rangle\}$ with $\zeta=0,\ldots,G$, where $|\zeta\rangle$ denotes a state with the domain wall at $\zeta+1/2$, i.e., $\sigma_{n>\zeta}=-$ and $\sigma_\zeta=+$. In this case, however, the transition probability $q_{G,\zeta}\sim(M-1)^\zeta$ increases exponentially with ζ. Hence we can ignore the other states except for the state with $\zeta=G$, that is, $|P\rangle$. Therefore, the autocorrelation function $C_H(t)$ for $|H\rangle$ is given by $C_P(t-2)$ multiplied by the transition probability $p_{G,G}$ from $|P\rangle$ to $|H\rangle$, which results in Eq. (14).

The scaling behavior of $C_P(t)$ and $C_H(t)$ is also confirmed via the numerical simulations. In Fig. 5, we show a plot of the autocorrelation function evaluated in the Ising spin chain of length $G\leqslant 400$ with $M=5$. As G increases, the decay follows the power law in t with the exponent -1. We also checked that the power-law scaling regime overlaps in the plots of $GC_P(t)$ and $G^2C_H(t)$ vs t.

It is easy to generalize the argument for the autocorrelation function to an arbitrary state $|\boldsymbol{\sigma}\rangle$ whose stationary state probability scales as $P_{\boldsymbol{\sigma}}^{\infty}\sim r^{-G}$. Since the perturbed domain size grows in time as $\xi\sim\ln t/\ln\kappa$, the value of $C_{\boldsymbol{\sigma}}(t)$ at $t\simeq\kappa^\xi$ is given by the stationary state probability for the spin configuration $\sigma_\xi\cdots\sigma_1$ in the chain of length ξ, i.e., $C_{\boldsymbol{\sigma}}(t\simeq\kappa^\xi)\simeq r^{-\xi}$. Eliminating ξ, one obtains that $C_{\boldsymbol{\sigma}}(t)\sim t^{-\alpha}$ with a state-dependent exponent $\alpha=\ln r/\ln\kappa$ [24]. The stationary state distribution is determined by the degree distribution. Therefore, we conclude that the nonuniversality (i.e.,

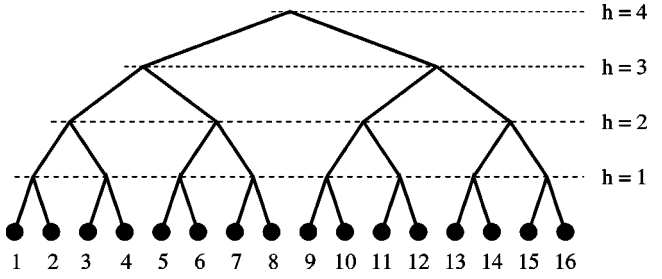

FIG. 6. Ultrametric space of 16 states.

state dependence) of the decay exponent is a consequence of the broad distribution of the degree in the underlying network.

V. ULTRAMETRIC DIFFUSION

In the preceding sections it turned out that the origin for the power-law decay of the autocorrelation functions is the hierarchical organization of the configuration space; the spins (or the random walker) overcome the dynamic barriers successively expanding the number of accessible configurations. We note that this phenomenon is very similar to the one observed in the diffusion in an ultrametric space [25,26]. In this section, we compare ultrametric diffusion with the random walk problem we have studied in this paper.

Consider a dynamical system with N states $a=1,2,\ldots,N$. The system in state a may perform transitions to any other state b with a transition probability w_{ab}. One can define the distance between two states as $d_{ab}=1/w_{ab}$ and thus provide the state space with a metric. If the transition probabilities satisfy the relation $1/w_{ab} \leqslant \sup(1/w_{ac};1/w_{bc})$ for all a, b, and c, the corresponding metric is called an *ultrametric* and the state space is an ultrametric space.

The simplest example of an ultrametric space is represented by a rooted tree generated as follows: We start from a single vertex at the Rth hierarchy and branch B vertices in the next $(R-1)$th hierarchy. Each of them branches into B vertices. It is repeated until one has $N=B^R$ vertices at the zeroth or bottom level. One then associates the vertices at the bottom level with the N states. The transition probabilities between two states are assigned to $w_{ab}=e^{-d\Delta}$, where $\Delta>0$ is a constant and d is the *hierarchical distance* between them, namely the hierarchy level of their common ancestor at the lowest level. It is easy to see that the transition probabilities satisfy the ultrametric relation, and thus an ultrametric space of N states is obtained. As an example, we illustrate in Fig. 6 the rooted tree with $R=4$ and $B=2$ for an ultrametric space of $N=16$ states. In this example, two states 1 and 7 have the common ancestors at the hierarchy level $h=3$ and 4, hence $w_{1,7}=e^{-3\Delta}$, while $w_{1,9}=e^{-4\Delta}$.

The autocorrelation function can be calculated exactly; see, e.g., Ref. [25]. The exact result is also understood with a simple scaling argument. Suppose that the system is in a state a initially. Since the transition probability to a state at the hierarchical distance ξ is given by $w=e^{-\Delta\xi}$ and there are $O(B^{\xi})$ such states, it takes $t_{\xi}\sim(B^{\xi}e^{-\Delta\xi})^{-1}$ time steps for the system to reach one of the states within the hierarchical distance ξ. Hence, the autocorrelation function at time $t\sim t_{\xi}$ is given by $C(t_{\xi})\sim B^{-\xi}$. Eliminating ξ, one obtains that the autocorrelation function decays algebraically as $C(t)\sim t^{-\alpha}$ with $\alpha=\ln B/(\Delta-\ln B)$. The power-law decay is valid for $\Delta>\ln B$, while the dynamics is unstable for $\Delta<\ln B$. At the marginal case, a stretched exponential decay $P(t)\sim e^{-(\ln B)t^{1/\gamma}}$ may occur when the transition probability decreases as $w(d)\sim d^{-\gamma}e^{-(\ln B)d}$ with the hierarchical distance [25,26].

Comparing the phenomenology, it is clear that the diffusion in the hierarchical network is essentially the same as the ultrametric diffusion. In both processes, the relaxation takes place by overcoming dynamic barriers successively and increasing associated length scale. The length scale corresponds to the perturbed domain size $\xi(t)\sim\ln t/\ln\kappa$ in the former, and to the hierarchical distance $\xi(t)\sim\ln t/\ln(\Delta-\ln B)$ in the latter. The length scale grows logarithmically in time, which is a consequence of the exponential increase of the dynamical barrier height.

Note, however, that the diffusion in the hierarchical network is not the ultrametric diffusion in a strict sense since the ultrametric relations are not valid. The configuration space of the Ising spin system has a tree structure with *all vertices* corresponding to physical states. The ultrametricity would hold only if vertices at the bottom hierarchy would represent physical states, see Figs. 2 and 6. Such a difference does not modify the ultrametric nature of the relaxation from the states $|A1\rangle$ and $|A2\rangle$, which are located at the end branch in the configuration space. On the other hand, the relaxations from $|H\rangle$ and $|P\rangle$, which are in the center of the configuration space tree, are influenced by the nonultrametricity. It is reflected in the G^{-1} and G^{-2} factors in the autocorrelation functions $C_P(t)$ and $C_H(t)$, respectively. Pseudoultrametric diffusion is also observed for the random walks on a tree structure [27] and on the one-dimensional lattice with hierarchically distributed dynamic barriers [28].

VI. SUMMARY

In summary, we have studied the random walk problem on the hierarchical network. The random walk problem on the network of $N=M^G$ nodes is mapped to a specially constrained dynamics of a M-state Potts spin chain of length G. Using the symmetry property, it is further mapped to a specially constrained dynamics of an Ising spin chain. From the analysis of the MFPT, it is shown that the characteristic relaxation time scales as $T\sim\kappa^G\sim N^z$ with $\kappa=M/(M-1)$ and $z=\ln\kappa/\ln M$. It is also shown that the autocorrelation function decays algebraically in time as $C_{\boldsymbol{\sigma}}(t)\sim t^{-\alpha_{\boldsymbol{\sigma}}}$ for $t\ll T$ with a nonuniversal (i.e., state-dependent) exponent $\alpha_{\boldsymbol{\sigma}}$. The power-law scaling behavior is closely related to the ultrametric diffusion. The exponent is given by $\alpha_{\alpha}=\ln r_{\boldsymbol{\sigma}}/\ln\kappa$ for a state $\boldsymbol{\sigma}$ whose stationary state probability is $P^{\infty}_{\boldsymbol{\sigma}}\sim r_{\boldsymbol{\sigma}}^{-G}$. The stationary state probability is determined from the degree of the corresponding nodes in the network. The broad distribution of the degree gives rise to the nonuniversality (state dependency) of the decay exponent.

The power-law decay of the autocorrelation functions appears in marked contrast to the stretched-exponential decay

in random networks [14] and in the small-world networks [16]. In order to investigate the origin of the emergence of the power-law scaling, we have also studied the random walks on the hierarchical networks with $M=2$. At $M=2$, the hierarchical network is not scale-free anymore. Nevertheless, we can use the same mapping to the Ising spin system with the configuration space of the same tree structure. So we can obtain the scaling behaviors of the relaxation time T using Eqs. (26) and (27), and of the autocorrelation function using the same scaling arguments: The relaxation time scales as $T\sim G^{\ln 2}\sim N^1$, and the autocorrelation functions decay algebraically in time with the *universal* (state-independent) exponent, i.e., $\alpha_H=\alpha_P=\alpha_A=1$. For $M=2$, the corresponding spin dynamics has the spin up-down symmetry. So, $C_H(t)$ and $C_P(t)$ decay in the same way as $C_H(t)=C_P(t)\sim t^{-1}/G$ with the same dependency on G. Finally, the scaling behavior of the relaxation time and the autocorrelation functions was confirmed numerically.

Comparing the results for $M=2$ and $M>2$, we conclude that the power-law scaling behavior of the relaxation time and the autocorrelation functions has its origin in the tree structure of the spin configuration space as shown in Fig. 2. We also conclude that the nonuniversality of the decay exponent for $M>2$ results from the scale-free degree distribution.

The hierarchical network itself does not have a tree structure. But, after the mapping, the random walk problem on the network reduces to that on the tree structure. In general, it is not known *a priori* whether such a mapping exists for an arbitrary network. It would be interesting to study the random walk problem on general networks in order to scrutinize the robustness of the power-law scaling behavior and the effect of the scale-free degree distribution on the relaxation dynamics. Such work is actually in progress.

We note that the very slow relaxation dynamics of the Ising chain representation of the random walk problem on the hierarchical network is due to the severe constraints of the dynamics imposed by the restrictions for possible transitions. Constrained dynamics in otherwise (for instance, thermodynamically) very simple models lead quite frequently to a slow or *glassy* dynamics [29], for which reason kinetically constrained models are often used as models for the dynamics in glasses and spin glasses. It is interesting to note that such a model also occurs in the context of diffusion in complex networks as we have demonstrated in this work.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); S.N. Dorogovtsev and J.F.F. Mendes, Adv. Phys. **51**, 1079 (2002).
[2] D.J. Watts and S.H. Strogatz, Nature (London) **393**, 440 (1998).
[3] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **401**, 130 (1999).
[4] S.N. Dorogovtsev, A.V. Goltsev, and J.F.F. Mendes, Phys. Rev. E **66**, 016104 (2002); F. Iglói and L. Turban, *ibid.* **66**, 036140 (2002).
[5] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001).
[6] E. Ravasz, A.L. Somera, D.A. Mongru, Z.N. Oltvai, and A.-L. Barabási, Science **297**, 1551 (2002); E. Ravasz and A.-L. Barabási, Phys. Rev. E **67**, 026112 (2003).
[7] L.A. Adamic, R.M. Lukose, A.R. Puniyani, and B.A. Huberman, Phys. Rev. E **64**, 046135 (2001).
[8] Q. Lv, P. Cao, E. Cohen, K. Li, and S. Shenker, in *Proceedings of the 16th International Conference on Supercomputing*, edited by K. Ebcioglu, K. Pingali, and A. Nicolau (ACM Press, New York, 2002), pp. 84–95; http://parapet.ee.princeton.edu/~sigm2002/papers/p258-lv.pdf
[9] N. Sarshar, V. P. Roychowdury, and P. O. Boykin, http://www.ee.ucla.edu/~nima/Publications/search_ITPTS.pdf
[10] R.D. Hughes, *Random Walks and Random Environments* (Clarendon, Oxford, 1995), Vol. 1.
[11] M.E.J. Newman, e-print cond-mat/0309045.
[12] J.D. Noh and H. Rieger, e-print cond-mat/0307719.
[13] H. Zhou, Phys. Rev. E **67**, 061901 (2003).
[14] A.J. Bray and G.J. Rodgers, Phys. Rev. B **38**, 11 461 (1988); I.A. Campbell, J. Phys. (France) Lett. **46**, L1159 (1985); I.A. Campbell, J.M. Flesselles, R. Jullien, and R. Botet, J. Phys. C **20**, L47 (1987); P. Jund, R. Jullien, and I. Campbell, Phys. Rev. E **63**, 036131 (2001).
[15] S.A. Pandit and R.E. Amritkar, Phys. Rev. E **63**, 041104 (2001).
[16] E. Almaas, R.V. Kulkarni, and D. Stroud, Phys. Rev. E **68**, 056105 (2003).
[17] S. Jespersen, I.M. Sokolov, and A. Blumen, Phys. Rev. E **62**, 4405 (2000).
[18] J. Lahtinen, J. Kertész, and K. Kaski, Phys. Rev. E **64**, 057105 (2001).
[19] B. Tadić, Eur. Phys. J. B **23**, 221 (2001).
[20] B. Tadić, e-print cond-mat/0310014.
[21] S. Jung, S. Kim, and B. Kahng, Phys. Rev. E **65**, 056101 (2002).
[22] S.N. Dorogovtsev, A.V. Goltsev, and J.F.F. Mendes, Phys. Rev. E **65**, 066122 (2002).
[23] J.D. Noh, Phys. Rev. E **67**, 045103(R) (2003).
[24] It is assumed that the constant r does not depend on the length scale. So, the power-law scaling in time will be observed for the states where spins are distributed rather uniformly. A complex behavior would emerge otherwise.
[25] A.T. Ogielski and D.L. Stein, Phys. Rev. Lett. **55**, 1634 (1985).
[26] G. Paladin, M. Mézard, and C. de Dominicis, J. Phys. (France) Lett. **46**, L985 (1985).
[27] M. Schreckenberg, Z. Phys. B: Condens. Matter **60**, 749 (1985).
[28] B.A. Huberman and M. Kerszberg, J. Phys. A **18**, L331 (1985).
[29] F. Ritort and P. Sollich, Adv. Phys. **52**, 219 (2003).