

Ground State Properties of Fluxlines in a Disordered Environment

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A new numerical method to calculate exact ground states of multifluxline systems with quenched disorder is presented, which is based on the minimum cost flow algorithm from combinatorial optimization. We discuss several models that can be studied with this method including their specific implementations, physically relevant observables and results: (1) The N -line model with N fluxlines (or directed polymers) in a d -dimensional environment with point and/or columnar disorder and hard or soft-core repulsion; (2) the vortex glass model for a disordered superconductor in the strong screening limit; and (3) the sine-Gordon model with random phase shifts in the strong coupling limit. [S0031-9007(98)07658-3]

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Dirty type-II superconductors in a magnetic field are the most intensively studied representatives of elastic manifolds in a disordered environment [1]. Their paradigmatic description consists in an ensemble of magnetic fluxlines (or vortexlines) interacting strongly with point and/or columnar defects and among themselves. This complicated multiline situation is usually reduced to the study of a single line, a directed polymer in a random medium [2], a problem that possesses deep connection also to nonequilibrium fluctuations of moving interfaces [3].

Here we are going to present a new numerical method (in the spirit of other recent applications of combinatorial optimization tools in the physics of disordered systems [4]) by which the investigation of the full multiline situation becomes feasible. It will enable us to determine exact ground states (i.e., minimum energy configurations) of these systems in polynomial time. Since the low temperature physics of fluxlines in a random environment is dominated by disorder effects these ground state calculations will enable us to make various statements about possibly glassy features, for instance the roughness of multiline systems, the stiffness of vortex or gauge glass models, and the displacement-displacement correlations in random phase models.

To introduce the notation and to set the stage of the theoretical models we consider we start with a simple but nontrivial (and hence heavily discussed [2]) example: the so-called 1-line problem, which consists in determining the minimum energy configuration of a single (magnetic) fluxline or a directed polymer (for a 111 lattice) in a disordered environment. The lattice version of this model is given by the Hamiltonian (or energy function)

$$H(\mathbf{x}) = \sum_{(ij)} e_{ij} \cdot x_{ij}, \quad (1)$$

where $\sum_{(ij)}$ is a sum over all *bonds* (ij) joining site i and j of a d -dimensional lattice, e.g., a rectangular ($L^{d-1} \times H$) lattice, with periodic boundary conditions (b.c.) in $d-1$ space directions and free b.c. in one direction. The bond

energies $e_{ij} \geq 0$ are quenched random variables that indicate how much energy it costs to put a segment of fluxline on a specific bond (ij). The fluxline configuration \mathbf{x} ($x_{ij} \geq 0$), also called a *flow*, is given by specifying $x_{ij} = 1$ for each bond (ij), which is occupied by the fluxline and $x_{ij} = 0$ otherwise. For the configuration to form *lines* on each site of the lattice all incoming flow should balance the outgoing flow, i.e., the flow is divergence free

$$\nabla \cdot \mathbf{x} = 0, \quad (2)$$

where $\nabla \cdot$ denotes the lattice divergence. The line(s) can enter the system anywhere on one free surface and leave it on the opposite surface. Therefore, we attach all sites of one free boundary to an extra site (via energetically neutral arcs, $e = 0$), which we call the source s , and the other side to another extra site, the target, t as indicated in Fig. 1a. Now one can push one line through the system by inferring that s has a source strength of $+1$ and that t has a sink strength of -1 , i.e.,

$$(\nabla \cdot \mathbf{x})_s = +N \quad \text{and} \quad (\nabla \cdot \mathbf{x})_t = -N, \quad (3)$$

with $N = 1$. Thus, the 1-line problem consists in minimizing the energy (1) by finding a flow \mathbf{x} in the network (the lattice plus the two extra sites s and t), fulfilling the constraints (2) and (3).

The solution of this problem is equivalent to finding the shortest path from s to t , where distances between two lattice sites are identified with the energies e_{ij} , which can either be done with Dijkstra's algorithm from combinatorial optimization [5] or by equivalent methods better known to physicists: the transfer matrix method [6].

Since this 1-line problem has been extensively studied [2] we directly proceed to its full generalization to N fluxlines, which has, apart from a special situation in $d = 2$ [7], never been treated in the literature before. The reason is simple: Whereas two lines ($N = 2$) are still tractable [8], the transfer matrix method fails to work efficiently for an increasing number of lines since its complexity grows exponentially with N . Since it is the dense limit

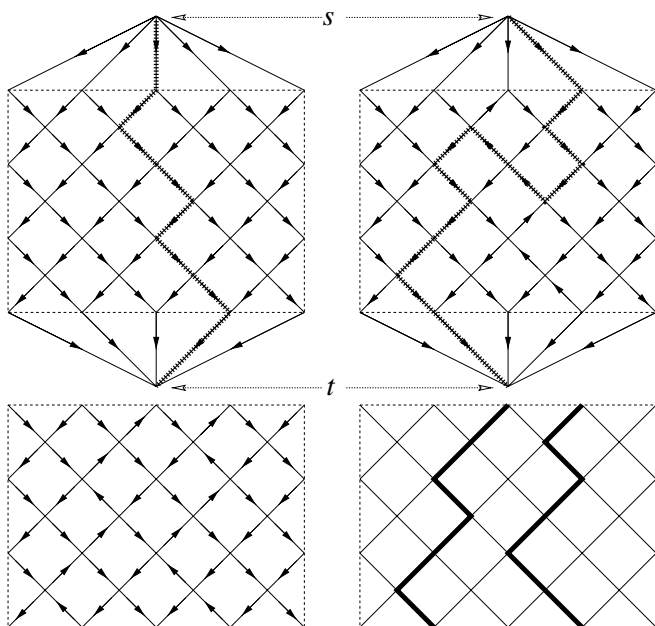


FIG. 1. Sketch of the residual network with shortest path before putting in the first fluxline (top left) and after the update for one fluxline (top right). Bottom: Residual network after second iteration (left) and the actual optimal fluxline configuration (right). Note that this case is a nontrivial case when the first fluxline has to be deformed in order to achieve the optimal two-line configuration.

$N = \rho L^{d-1}$ with ρ of the order of 1 which is expected to contain new physics an algorithm that solves this problem in polynomial time as the one we are now going to present, is highly desirable.

The N -line problem again consists in minimizing (1) in such a way that (2) and (3) are fulfilled, now with an arbitrary value for N . Physically one has to take into account a repulsive interaction between the fluxlines, for instance a hard-core repulsion, which can be modeled by inferring that $x_{ij} \in \{0, 1\}$, i.e., that only a segment of one single fluxline can pass through an arc [9]. It is also possible to apply our method to a situation with soft-core repulsion, which we discuss below. The problem is now formulated in such a way that it is identical to a minimum cost flow problem in combinatorial optimization [5].

Since the 1-line problem can be solved by finding a shortest path the intuitive idea to solve the N -line problem would be to search successively N shortest path, i.e., by adding one fluxline after the other to the system. However, adding a fluxline to an existing fluxline configuration might necessitate redirecting one or more fluxlines, as indicated in Fig. 1. This at first sight formidable task is elegantly solved by the so-called successive shortest path algorithm for minimum cost flow problems [5].

The first key ingredient is that one does not work with the original network but with the residual network corresponding to the actual fluxline configuration, which contains also the information about possibilities to send flow backwards (now with energy $-e_{ij}$ since one wins energy by reducing x_{ij}), i.e., to modify the actual flow.

Suppose that we put one fluxline along a shortest path $P(s, t)$ from s to t , which means that we set $x_{ij} = 1$ for all arcs on the path $P(s, t)$. Then the residual network is obtained by reversing all arcs and inverting all energies along this path, indicating that here we cannot put any further flow in the forward direction (since we assume hard-core interaction, i.e., $x_{ij} \leq 1$), but can send flow backwards by reducing x_{ij} on the forward arcs by one unit. This procedure is sketched in Fig. 1.

The second key ingredient is the introduction of a so-called potential φ that fulfills the relation

$$\varphi(j) \leq \varphi(i) + e_{ij} \quad (4)$$

for all arcs (ij) in the residual network, indicating how much energy $\varphi(j)$ it would *at least* take to send one unit of flow from s to site j , if it would cost an energy $\varphi(i)$ to send it to site i . With the help of these potentials one defines the reduced costs

$$c_{ij}^{\varphi} = e_{ij} + \varphi(i) - \varphi(j) \geq 0. \quad (5)$$

The last inequality, which follows from the properties of the potential φ (4) actually ensures that there is no loop \mathcal{L} in the current residual network (corresponding to a flow \mathbf{x}) with negative total energy, since $\sum_{(ij) \in \mathcal{L}} e_{ij} = \sum_{(ij) \in \mathcal{L}} c_{ij}^{\varphi}$, implying that the flow \mathbf{x} is optimal [5].

It is important to note that the inequality (4) is reminiscent of a condition for shortest path distances $d(i)$ from s to all sites i with respect to the energies e_{ij} : they have to fulfill $d(j) \leq d(i) + e_{ij}$. Thus, one uses these distances d to construct the potential φ when putting one fluxline after the other into the network:

We start with the empty network (zero fluxlines) $\mathbf{x}^0 = 0$, which is certainly an optimal flow for $N = 0$, and set $\varphi = 0$, $c_{ij}^{\varphi} = e_{ij}$. Next, let us suppose that we have an optimal $N - 1$ -line configuration corresponding to the flow \mathbf{x}^{N-1} . The current potential is φ^{N-1} , the reduced costs are $c_{ij}^{N-1} = e_{ij} + \varphi^{N-1}(i) - \varphi^{N-1}(j)$ and we consider the residual network G_c^{N-1} corresponding to the flow \mathbf{x}^{N-1} with the reduced costs $c_{ij}^{N-1} \geq 0$. The iteration leading to an optimal N -line configuration x_{ij}^N is as follows:

(1) Determine a shortest path $P(s, t)$ with respect to the reduced costs c_{ij}^{N-1} from s to t in the residual network G_c^{N-1} .

(2) For all site on $P(s, t)$ let $d(i)$ be the shortest path distance from s to i . For these update the potentials: $\varphi^N(i) = \varphi^{N-1}(i) + d(i) - d(t)$.

(3) To obtain x_{ij}^N increase (decrease) by one unit the flow variables x_{ij}^{N-1} on all forward (backward) arcs (ij) on the shortest path $P(s, t)$.

Note that due to the fact that the numbers $d(i)$ are shortest distances one has again $c_{ij}^N \geq 0$, i.e., the flow \mathbf{x}^N is indeed optimal. The complexity of this iteration is the same as that of Dijkstra's algorithm for finding shortest paths in a network, which is $\mathcal{O}(M^2)$ in the worst case (M is the number of nodes in the network). We find, however, for the cases we consider (d -dimensional

lattices) it roughly scales linear in $M = L^d$. Thus, for N fluxlines the complexity of this algorithm is $\mathcal{O}(NL^d)$.

In Fig. 2 we show a number of optimal fluxline configurations with varying line density obtained with our algorithm for the N -line problem on a $L \times L$ square lattice. We would like to emphasize the algorithm works for fluxlines in arbitrary dimension, even for general graphs. One observes that the roughness $[\Delta w]_{\text{av}}$ of the fluxlines decreases systematically with increasing line density from the single line limit ($N = 1$) $[\Delta w]_{\text{av}} \sim L^\zeta$ with the roughness exponent $\zeta = 2/3$ to the dense limit ($N = L$ in $d = 2$) with no roughness at all, $[\Delta w]_{\text{av}} = 0$. For a finite fluxline density $\rho = N/L$ each fluxline is essentially free up to a length $\xi_{\parallel} = \xi_{\perp}^{1/\zeta}$ with $\xi_{\perp} = 1/\rho$. Thus one expects for $d = 2$ the finite size scaling form

$$[\Delta w]_{\text{av}} \sim L^\zeta \tilde{w}(H\rho^\nu) \quad \text{with } \nu = 1/\zeta, \quad (6)$$

where H is the height and \tilde{w} is a scaling function with $\tilde{w}(x) \rightarrow \text{const}$ for $x \rightarrow 0$. In Fig. 3 we show a corresponding scaling plot for the data obtained with our algorithm for $d = 2$ (and $H = L$) [11].

Before we proceed we would like to point out that the Hamiltonian (1) is general enough to describe various physically interesting situations. By an appropriate definition of the energies e_{ij} one can easily model columnar defects [10], disorder induced melting of a fluxline lattice (e.g., the Abrikosov lattice) and depinning transitions.

Next we consider soft-core repulsion, which can be modeled by allowing a multiple occupancy of a bond ($x_{ij} = 0, 1, 2, \dots$) but punish high fluxline densities with an energy $\tilde{e}_{ij}(x_{ij})$ increasing faster than linear with the number of flux units x_{ij} on the bond (ij). Thus the N -line problem with soft repulsion consists in minimizing

$$\tilde{H}(\mathbf{x}) = \sum_{(ij)} \tilde{e}_{ij}(x_{ij}), \quad (7)$$

under the constraints (2) and (3). The local energy functions \tilde{e}_{ij} can be chosen arbitrarily for each bond (ij), however, they have to be *convex* as for instance $\tilde{e}_{ij}(x_{ij}) = k_{ij} \cdot x_{ij}^n$ with $n \geq 1$ arbitrary. The energies e_{ij} have now to be

replaced by the quantity $\tilde{e}_{ij}(x_{ij} + 1) - \tilde{e}_{ij}(x_{ij})$, which is the energy needed to increase the flow x_{ij} on arc (ij) by one unit. Since it depends on the current flow \mathbf{x} the convexity of \tilde{e}_{ij} is needed to ensure that the reduced costs fulfill the inequality $c_{ij}^N \geq 0$ (5) also after the flow modification. Whereas with hard-core repulsion it was only possible to put $N = L^d$ fluxline into the system, the fluxline density can now be arbitrarily high and an interplay between the repulsion and the disorder effects lead to a much richer phenomenology [11].

Up to now we considered situations in which the fluxlines are put into the system via an explicit external source. We now present a model in which fluxlines are generated *inside* the system: namely the vortex representation of the gauge glass model [12] with strong screening, which is given by the Hamiltonian [13]

$$H_V = \sum_{(ij)} (x_{ij} - b_{ij})^2. \quad (8)$$

Here $\sum_{(ij)}$ is a sum over all bonds of a simple cubic lattice ($d = 3$) with periodic b.c. in all directions and we do *not* have external source nodes subjected to condition (3). The x_{ij} are the integer flow variables that have to fulfill the divergence free condition (2), and the b_{ij} are quenched random variables that are *real* numbers. They can be arbitrary, however in the gauge glass they fulfill a divergence free condition $\nabla \cdot \mathbf{b} = 0$ since they represent a magnetic field derived from a quenched random vector potential ($\mathbf{b} = \nabla \times \mathbf{A}$). Without the constraint (2) the optimal solution would simply be given by choosing x_{ij} to be the closest integer to b_{ij} . This solution fulfills $c_{ij}^\varphi \geq 0$ with $\varphi = 0$, where the costs are chosen as for the convex flow problem (7). Since it violates the constraints (2) one has either excess or deficit flow entering or leaving individual sites, which one has to remove. Instead of sending flow from one particular source node s to a target t as in the fluxline problem, one now sends flow from excess to deficit sites along shortest paths using the iteration described above. In this way one successively removes the violations of constraint (2) by

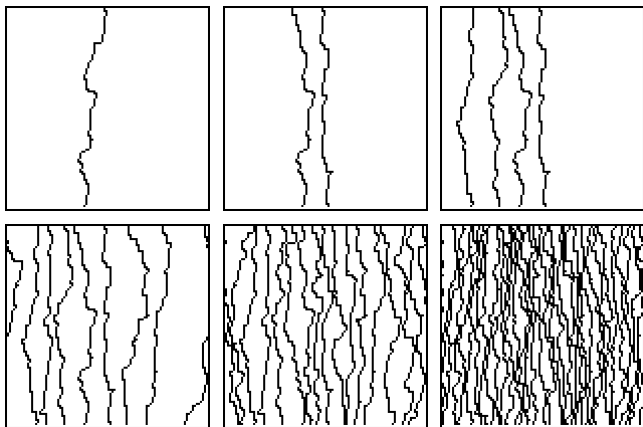


FIG. 2. Optimal fluxline configuration for one particular 2D sample of linear size $L = 100$. The number of fluxlines, N , is 1,2,4 (top) and 8,16,32 (bottom).

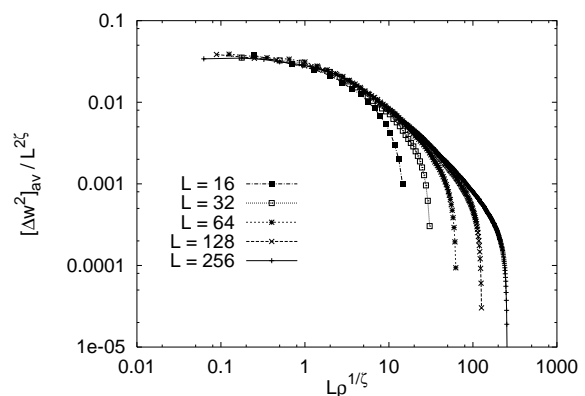


FIG. 3. Scaling plot for the roughness (per line) in the two-dimensional multifluxline system; cf. (6). The data are averaged over 1000 different disorder configurations.

keeping the reduced cost optimality $c_{ij}^{\circ} \geq 0$ all the time, which guarantees the optimality of the flow at the end of the iteration.

The physically most interesting question in the context of the model Hamiltonian (8) concerns the existence of a superconducting glass phase. This can be studied via domain wall renormalization group methods [14], by which one determines the scaling behavior of low lying excitation ΔE on the length scale L . Such an excitation in the vortex representation (8) is a loop (closed fluxline) with an area proportional to L^2 , which can be realized by an extra fluxline (on the background of the true ground state) winding once around the 3D torus in one direction (note that we have periodic b.c. in all directions). Details of this procedure can be found in [15]; here we give only the result: $\Delta E \sim L^{\theta}$ with $\theta = -0.95 \pm 0.03$. From this we can draw two conclusions: (a) since it is clearly negative there is no superconducting (vortex) glass phase at nonvanishing temperature, and (b) the thermal correlation length diverges only with $T \rightarrow 0$ as $\xi_{\text{th}} \sim T^{-\nu}$ with an exponent $\nu = 1/|\theta| \sim 1.05 \pm 0.03$, which is in agreement with Ref. [13].

Finally, as our last application of our algorithm we discuss the Sine-Gordon model with random phase shifts, which in two dimensions is a model for a fluxline array [16]

$$H_{\text{RP}} = \sum_{(ij)} |u_i - u_j|^m - \lambda \sum_i \cos[2\pi(u_i - \beta_i)], \quad (9)$$

where u_i are (real) displacements in a fluxline array, $\beta_i \in [0, 1]$ (quenched) random phase shifts, (ij) nearest neighbor pairs of a d -dimensional lattice, $m \geq 1$ is some number [usually it is $m = 2$, by which the first term in (9) becomes an elastic energy] and λ a coupling parameter. In the limit of strong coupling $\lambda \rightarrow \infty$ one enforces $u_i = \beta_i + n_i$, where n_i is an integer, and the Hamiltonian reads in these variables $H'_{\text{RP}} = \sum_{(ij)} |n_i - n_j - \beta_{ij}|^m$, where $\beta_{ij} = \beta_i - \beta_j$. In $d = 2$ this model is equivalent to a model for a crystalline surface on a random substrate [17], for which the variables n_i are interpreted as integer height variables and β_i as substrate heights between 0 and 1.

By introducing the variables $x_{ij} = n_i - n_j$ one observes that the form of the Hamiltonian is a special case of the convex cost functions considered above (7) for which the ground state can again be computed with the algorithm we have presented. However, now one has to consider a simply connected topology for the underlying network (e.g., *not* the torus) since we have to reconstruct the variables n_i from their differences x_{ij} . Thus we assume free or fixed b.c. for the lattice. In two dimensions $d = 2$ we studied the correlation function $G(r) = [(n_i - n_{i+r})^2]_{\text{av}}$ for various strengths of the nonlinearity m , and found that $G(r)$ increases stronger than $\log(r)$ with the distance, possibly like $\log^2(r)$, indicating a superrough low temperature phase, as for $m = 2$ [18,19] and $m = 1$ [7].

To summarize we have presented various applications of a successive shortest path algorithm for disordered systems

containing many fluxlines or directed polymers with short range repulsion and derived a number of new results. It would be of high interest to think about generalization to fluxlines with long range (like Coulomb) interaction. [20].

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