Long-Range Correlations in the Nonequilibrium Quantum Relaxation of a Spin Chain

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We consider the nonstationary quantum relaxation of the Ising spin chain in a transverse field of strength h. Starting from a homogeneously magnetized initial state the system approaches a stationary state by a process possessing quasi-long-range correlations in time and space, independent of the value of h. In particular, the system exhibits aging (or lack of time-translational invariance on intermediate time scales) although no indications of coarsening are present.

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Nonequilibrium dynamical properties of quantum systems have been of interest recently, experimentally and theoretically. Measurements on magnetic relaxation at low temperatures show deviations from the classical exponential decay [1], which was explained by the effect of quantum tunneling. On the theoretical side, among others, integrable [2] and nonintegrable models [3] were studied in the presence of energy or magnetic currents, as well as the phenomena of quantum aging in systems with long-range [4] and short-range interactions [5].

Here we pose a different question: Consider a quantum mechanical interacting many body system described by a Hamilton operator $\hat{H}$ without any coupling to an external bath, which means that the system is closed. Suppose the system is prepared in a specific state $|\psi_0\rangle$ at time $t = 0$, which is *not* an eigenstate of the Hamiltonian $\hat{H}$. Then we are interested in the natural quantum dynamical evolution of this state which is described by the Schrödinger equation and is formally given by

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi_0\rangle. \tag{1}$$

Obviously the energy $E = \langle\psi_0|\hat{H}|\psi_0\rangle$ is conserved. In particular, we want to study the time evolution of the expectation value $A(t)$ of an observable $\hat{A}$ or the two-time correlation function $C_{AB}(t_1, t_2)$ of two observables $\hat{A}$ and $\hat{B}$, defined by

$$\begin{aligned} A(t) &= \langle\psi_0|\hat{A}_H(t)|\psi_0\rangle, \\ C_{AB}(t_1, t_2) &= \langle\psi_0|\{\hat{A}_H(t_1)\hat{B}_H(t_2)\}_S|\psi_0\rangle, \end{aligned} \tag{2}$$

where $\hat{A}_H(t) = \exp(+i\hat{H}t)\hat{A}\exp(-i\hat{H}t)$ is the $\hat{A}$ in the Heisenberg picture (with $\hbar$ set to unity) and $\{\hat{A}\hat{B}\}_S = 1/2(\hat{A}\hat{B} + \hat{B}\hat{A})$ is the symmetric product of two operators.

One should emphasize that in such a situation one does *not* expect time translational invariance to hold, which would manifest itself in, for instance, $A(t) = A_0 = \text{const}$ and $C_{AB}(t_1, t_2) = C_{AB}(t_1 - t_2)$. There will be a transient regime in which these relations are violated and, depending on the complexity of the system, this *nonequilibrium* regime will extend over the whole time axis. Then we denote it as *quantum aging,* as it can be observed, e.g., for the Universe, which is (most probably) a closed system.

To be concrete we consider the prototype of an interacting quantum system, the Ising model in a transverse field in one dimension defined by the Hamiltonian

$$H = -\frac{1}{2}\left[\sum_{l=1}^{L-1}\sigma_l^x\sigma_{l+1}^x + h\sum_{l=1}^{L}\sigma_l^z\right], \tag{3}$$

where $\sigma_l^{x,z}$ are spin-1/2 operators on site l. We consider initial many body states that are eigenstates either of all local σ_l^x or of all local σ_l^z operators. We will mainly consider fully magnetized initial states, either in the x or the z direction, which we denote with $|x\rangle$ and $|z\rangle$ and which obey $\sigma_l^x|x\rangle = +|x\rangle$ and $\sigma_l^z|z\rangle = +|z\rangle$.

In passing we note that one obtains the zero temperature *equilibrium* situation by choosing the ground state of the Hamiltonian (3) as the initial state. This ground state has a quantum phase transition at $h = 1$ from a paramagnetic ($h > 1$) to a ferromagnetic ($h < 1$) phase. The former has long-range order (LRO) along the z direction; the latter has spontaneous symmetry breaking LRO along x. Moreover, the nonzero temperature ($T > 0$) *equilibrium* relaxation of (3) has been considered in [6] corresponding to an *open* system coupled to a heat bath in the stationary state, which is not related to the nonstationary *closed* system we consider here.

The expectation values and correlation functions we are interested in involve the spin operators σ_l^x and σ_l^z. To compute them, we express the Hamiltonian (3) in terms of fermion creation (annihilation) operators [7,8] η_q^+ (η_q)

$$H = \sum_q \epsilon_q\left(\eta_q^+\eta_q - \frac{1}{2}\right). \tag{4}$$

The energy of modes, ϵ_q, $q = 1, 2, \ldots, L$ are given by the solution of the following set of equations:

$$\begin{aligned} \epsilon_q\Psi_q(l) &= -h\Phi_q(l) - \Phi_q(l+1), \\ \epsilon_q\Phi_q(l) &= -\Psi_q(l-1) - h\Psi_q(l), \end{aligned} \tag{5}$$

and we use free boundary conditions which implies for the components $\Phi_q(L+1) = \Psi_q(0) = 0$. The spin operators can then be expressed by the fermion operators as

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$$\sigma_l^x = A_1 B_1 A_2 B_2 \dots A_{l-1} B_{l-1} A_l \,, \qquad \sigma_l^z = -A_l B_l \,, \tag{6}$$

with

$$A_i = \sum_{q=1}^{L} \Phi_q(i)\,(\eta_q^+ + \eta_q)\,, \qquad B_i = \sum_{q=1}^{L} \Psi_q(i)\,(\eta_q^+ - \eta_q)\,, \tag{7}$$

and the time evolution of the spin operators follows from the time dependence of the fermion operators: Inserting $\eta_q^+(t) = e^{it\epsilon_q}\eta_q^+$, $\eta_q(t) = e^{-it\epsilon_q}\eta_q$ into Eq. (7) one obtains

$$A_l(t) = \sum_k [\langle A_l A_k \rangle_t A_k + \langle A_l B_k \rangle_t B_k]\,, \qquad B_l(t) = \sum_k [\langle B_l A_k \rangle_t A_k + \langle B_l B_k \rangle_t B_k]\,, \tag{8}$$

with the time-dependent contractions

$$\langle A_l A_k \rangle_t = \sum_q \cos(\epsilon_q t)\Phi_q(l)\Phi_q(k)\,, \quad \langle A_l B_k \rangle_t = \langle B_k A_l \rangle_t = i\sum_q \sin(\epsilon_q t)\Phi_q(l)\Psi_q(k)\,, \quad \langle B_l B_k \rangle_t = \sum_q \cos(\epsilon_q t)\Psi_q(l)\Psi_q(k)\,. \tag{9}$$

For general position of the spin, $l = O(L/2)$, one finds simple formulas for the expectation values and correlation functions involving σ_l^z operators, whereas the calculation of those involving σ_l^x operators is a difficult task and the final result is complicated [9,10]. However, both the surface-spin autocorrelations and the end-to-end correlations are given in quite simple form, both for the equilibrium [8] and for the nonequilibrium case.

First we study the x-end-to-end correlations defined by

$$C_L^{x,\psi}(t) = \langle \psi_0 | \{\sigma_1^x(t)\sigma_L^x(t)\}_S | \psi_0 \rangle\,, \tag{10}$$

which contain information about the existence or the absence of magnetic order in the x direction. The single time t at which the correlations between the two spins are measured indicates the age of the system after preparation. For the fully ordered initial state $|\psi_0\rangle = |x\rangle$ we obtain

$$C_L^{x,x}(t) = \langle A_1 A_1 \rangle_t \langle B_L B_L \rangle_t + |\langle A_1 B_L \rangle_t|^2, \tag{11}$$

where the first term on the right-hand side of Eq. (11) is the product of surface magnetizations: $\overline{m}_1 = \langle x|\sigma_1^x(t)\,|x\rangle = \langle A_1 A_1 \rangle_t$ and $\overline{m}_L = \langle x|\sigma_L^x(t)\,|x\rangle = \langle B_L B_L \rangle_t$. In the next paragraph we show that $\lim_{t\to\infty} |\langle A_1 B_L \rangle_t|^2 = 0$. Therefore $\lim_{L,t\to\infty} C_L^{x,x}(t) = \overline{m}_1^2$ and the stationary state, starting with $|x\rangle$, has long-range order for $h < 1$ with $\overline{m}_1 = |\Phi_1(1)|^2 = 1 - h^2$. Thus the surface order parameter, $\overline{m}_1$, vanishes continuously at the transition point, $h_c = 1$, with a nonequilibrium exponent, $\beta_1^{\mathrm{ne}} = 1$.

The *connected* correlations are defined via Eq. (2) as $\tilde{C}_{AB}(t_1, t_2) = C_{AB}(t_1, t_2) - A(t_1)B(t_2)$. The time dependence of the connected end-to-end correlations $\tilde{C}_L^{x,x}(t) = |\langle A_1 B_L \rangle_t|^2 = |\sum_q \sin(\epsilon_q t)|\Phi_q(1)|^2(-1)^q|^2$ is a result of interference effects due to an interplay of length, L (via the difference in the excitation energies: $\epsilon_q - \epsilon_{q-1} \sim 1/L$), and time, t. Evidently they vanish both for small ($t \ll L$) and large ($t \gg L$) times, in the latter case due to random phase factors. For intermediate time scales we obtained through a numerical analysis of the formula the following features of the connected correlations which can be read from Fig. 1: (i) They are zero for times smaller than a time $\tau_h(L)$ which is equal to the system size L for $h \geq 1$ and increases monotonically with decreasing h for $h < 1$. (ii) At $t = \tau_h(L)$ a jump occurs to a value that decreases algebraically with L:

$$\tilde{C}_{\max}^{x,x}(L) = \tilde{C}_L^{x,x}(t = \tau_h(L)) \propto L^{-a}, \tag{12}$$

with $a = 2/3$ for $h = 1$ and $a = 1/2$ for $h > 1$ [11]. (iii) For $t \geq \tau_h(L)$ the correlations decay slower than exponentially as can be seen from the figure. (iv) For $t = 3\tau_h(L)$ again a sudden jump occurs as for $t = \tau_h(L)$ followed by a slightly slower oscillatory decay. (v) This pattern is repeated for time $t = 5\tau_h(L), 7\tau_h(L), \dots$ but gets progressively smeared out by oscillations.

These features can be interpreted as follows: the elementary (tunnel) processes of the quantum dynamics of the Hamiltonian (3) are spin flips induced by the transverse field operator σ_l^z. In this picture two spins can act only coherently and thus give a contribution to the connected correlation function if the information about such a spin flip process reaches the two spins simultaneously.

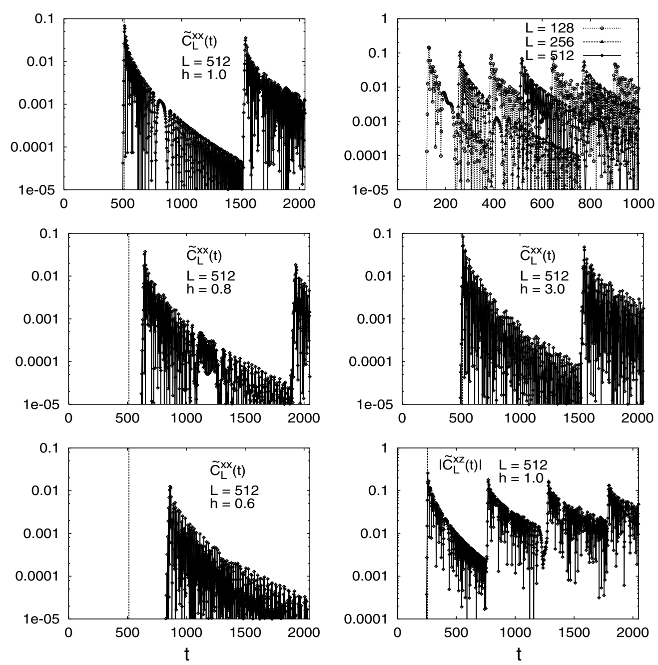

FIG. 1. Connected end-to-end correlations $\tilde{C}_L^{xx}$ (and $\tilde{C}_L^{xz}$ for bottom left) with fixed system size L and field h as a function of the time t calculated with Eqs. (11) and (13). The left column shows data for decreasing field strength $h = 1.0$, 0.8, 0.6; the broken vertical line is at $t = 512 = L$. The upper right figure shows $\tilde{C}_L^{xx}$ for different system sizes at $h = 1.0$, the middle right plot shows $\tilde{C}_L^{xx}$ at $h = 3.0$, and the lower right figure shows the modulus of $\tilde{C}_L^{xz}$ for $h = 1$. Here the broken vertical line is at $t = 256 = L/2$. For the interpretation, see text.

Feature i tells us that signals generated in the center of the system travel with a speed proportional to $L/\tau_h(L)$ to the boundary spins and reaches both simultaneously. At this moment $\tilde{C}_L^{x,x}(t)$ jumps to its maximum (see ii). After this, this signal is superposed by other more incoherent signals (see iii). However, the strongest initial signal is reflected at both boundaries and reaches the opposite boundary spins simultaneously again at time $t = 3\tau_h(L)$ (see iv), and so on. More and more incoherent processes occur in the meantime, resulting in feature v.

A similar behavior can be observed for the end-to-end correlations when starting with the state $|z\rangle$, which is

$$C_L^{x,z}(t) = \sum_k (\langle A_1 B_k\rangle_t \langle B_L A_k\rangle_t - \langle A_1 A_k\rangle_t \langle B_L B_k\rangle_t). \tag{13}$$

The only difference in the behavior of $C_L^{x,x}(t)$ reported above is (a) its long time limit vanishes for all values of h and (b) $\tau_h(L)$, i.e., the earliest time at which the two boundary spins are correlated is only half as big as in the previous case. Obviously it is easier to generate and to propagate spin flip signals when starting with a z state.

Next we study the *bulk* behavior of the expectation values and correlations involving σ_l^z operators. First we introduce the shorthand notation, $[D\tilde{D}]_{l,l'}^{t,t'} = \sum_k (\langle D_l B_k\rangle_t \langle \tilde{D}_{l'} A_{i(k)}\rangle_{t'} - \langle \tilde{D}_{l'} B_k\rangle_{t'} \langle D_l A_{i(k)}\rangle_t)$, with $D_l, \tilde{D}_l = A_l$ or B_l and $i(k) = k, (k+1)$ for $\psi = z, (x)$ and start with the nonequilibrium expectation value

$$e_l^{\psi}(t) = \langle\psi_0|\sigma_l^z(t)|\psi_0\rangle = [AB]_{l,l}^{t,t}. \tag{14}$$

We note that the equilibrium (i.e., ground state) expectation value, e_l^0, corresponds to the energy density in the two-dimensional classical Ising model and we use this terminology also in this nonequilibrium situation. At the transition point, $h = 1$, the contractions in Eq. (9) can be expressed in terms of Bessel functions, $J_\nu(x)$, as $\langle A_l A_k\rangle_t = \langle B_l B_k\rangle_t = (-1)^{l+k} J_{2l-2k}(2t)$ and $\langle A_l B_k\rangle_t = i(-1)^{l+k+1} J_{2l-2k+1}(2t)$ for $l = O(L/2)$. Equation (14) then yields $e_l^{\psi}(t) = 1/2 \pm J_1(4t)/4t$, where the $+$ $(-)$ sign refers to $\psi = z(x)$. Thus for long times the nonequilibrium energy density approaches the limit $\overline{e}_l^{\psi} = 1/2$ algebraically $\sim t^{-3/2}$. It can be shown that the decay exponent, 3/2, is universal; its value does not depend on the value of $0 < h < \infty$. Moreover, the stationary value of the energy density for a bulk spin can be calculated exactly for all initial states [9].

The two-spin nonequilibrium dynamical and spatial connected correlations can be expressed as

$$\begin{aligned}\tilde{C}_{l,l'}^{z,\psi}(t_1,t_2) &= \langle\psi_0|\{\sigma_l^z(t_1)\sigma_{l'}^z(t_2)\}_S|\psi_0\rangle \\ &= \langle A_l A_{l'}\rangle_{t_2-t_1}\langle B_l B_{l'}\rangle_{t_2-t_1} - \langle A_l B_{l'}\rangle_{t_2-t_1}\langle A_{l'} B_l\rangle_{t_2-t_1} - [AA]_{l,l'}^{t_1,t_2}[BB]_{l,l'}^{t_1,t_2} + [AB]_{l,l'}^{t_1,t_2}[BA]_{l,l'}^{t_1,t_2}.\end{aligned} \tag{15}$$

which is valid both for $|\psi_0\rangle = |x\rangle$ and $|\psi_0\rangle = |z\rangle$. In Fig. 2 we show the r dependence of $\tilde{C}^{z,z}(r,t)$ for various times t. We see that for fixed time t the correlations increase proportional to r^2 for distances $r \leq t$ to a maximum value $\tilde{C}_{\max}^{z,z}(t)$ at $r = 2t$, which decreases with time proportional to t^{-1}. For distances larger than $r = 2t$ they drop rapidly, faster than exponentially, to zero.

The latter two features correspond perfectly to what we observed also for the z-end-to-end correlations [see

The autocorrelation function ($l = l'$) for ($t_1 \leq t_2$) is generally nonstationary for intermediate times, $t_1/(t_2 - t_1) = O(1)$. In the limit $L \to \infty$ at $h = 1$ it can be expressed with Bessel functions via Eq. (15):

$$\tilde{C}_{l,l}^{z,\psi}(t_1,t_2) = J_0^2(2t_2 - 2t_1) - \frac{1}{4}[f(t_2+t_1) \pm g(t_2-t_1)], \tag{16}$$

where $f(x) = J_2(2x) + J_0(2x)$, $g(x) = J_2(2x) - J_0(2x)$, and the $+$ $(-)$ sign refers to $\psi = x(z)$. Thus we conclude that for intermediate times there is *aging* in the z-component autocorrelation function, contrary to what is reported in [5]. Asymptotically we have $\lim_{t_1\to\infty} C_{l,l}^{z,\psi}(t_1,t_2) = (\overline{e}^{\psi})^2$, and the connected two-time correlations depend only on the time difference, e.g., for $h = 1$ via Eq. (16) $\lim_{t_1\to\infty}\tilde{C}_{l,l}^{z,\psi}(t_1,t_2) = J_0^2(2[t_2 - t_1]) - \{J_1'(2[t_2 - t_1])\}^2$. For bulk spins this stationary correlation function decays algebraically as $\sim(t_2 - t_1)^{-2}$ for any value of h.

Next we consider the spatial connected equal-time correlations, $\tilde{C}^{z,\psi}(r,t)$, which follow from Eq. (15) with $t_1 = t_2 = t$ and $l = (L-r)/2$, $l' = (L+r)/2$. At the critical point, $h = 1$, in a similar way to the autocorrelation function one gets in the limit $L \to \infty$,

$$\tilde{C}^{z,\psi}(r,t) = \left[\frac{r}{2t} J_{2r}(4t)\right]^2 - \frac{r^2 - 1}{4t^2} J_{2r+1}(4t) J_{2r-1}(4t), \tag{17}$$

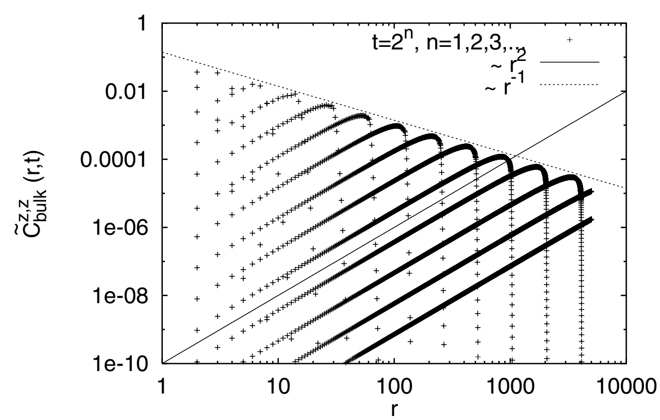

FIG. 2. Connected σ_z correlation function $\tilde{C}^{z,\psi}(r,t)$ at $h = 1$ given by the expression Eq. (17) for different times ($t = 2^n$, $n = 1, 2, 3, \ldots$ from left to right) in a log-log plot. The two straight lines indicate the initial r^2 dependence of $\tilde{C}^{z,\psi}(r,t)$ for fixed t as well as the r^{-1} dependence of the maximum value at $r = 2t$.

TABLE I. Power law dependencies of correlations on and behind the front.

		$h = 1$		$h > 1$	
	$\alpha_{\max}$	$\alpha = \alpha_{\max}$	$\alpha > \alpha_{\max}$	$\alpha = \alpha_{\max}$	$\alpha > \alpha_{\max}$
$\tilde{C}_L^{xx}(t = \alpha L)$	1	$L^{-2/3}$	L^{-1}	$L^{-1/2}$	L^{-1}
$\tilde{C}_L^{xz}(t = \alpha L)$	1/2	$L^{-1/4}$	$L^{-1/2}$	...	...
$C_L^{z\psi}(t = \alpha L)$	1/2	$L^{-5/4}$	L^{-1}	$L^{-5/8}$	L^{-1}
$\tilde{C}^{z\psi}(t = \alpha r)$	1/2	$r^{-4/3}$	r^{-1}	$r^{-2/3}$	r^{-1}

Eq. (13)]: spins that are separated by a distance r can be correlated only after the first signal from spin flip processes in between them reaches simultaneously the two spins, i.e., for times t larger than $r/2$ (for $h = 1$ and $|\psi_0\rangle = |z\rangle$. The first feature, that correlations for distances smaller than $2t$ are diminished only algebraically, is different from the faster decay of end-to-end correlations and is characteristic for bulk spins. For $r \leq 2t$ the correlation function $\tilde{C}^{z,\psi}(r,t)$ obeys the characteristic scaling form

$$\tilde{C}^{z,\psi}(r,t) = t^{-1} g(r/t) \tag{18}$$

with $g(x) \propto x^2$ for $x \ll 1$. The scaling parameter r/t appearing in the scaling function $g(x)$ is reminiscent of the fact that space and time scales are connected linearly at the critical point in the transverse Ising chain since the dynamical exponent is $z = 1$. Away from the critical point we have to evaluate our expressions [9] for $\tilde{C}^{z,\psi}(r,t)$ numerically. The results show similar features as the case $h = 1$ and will be presented elsewhere [9].

For completeness we finally list our results for the maximum value for connected spin-spin correlations since they decay algebraically with various new exponents; a detailed derivation will be given in [9]. We confine ourselves to $h \geq 1$ since here the time τ_h of maximum correlation is fixed, whereas for $h < 1$ the value of τ_h depends on h and has to be determined numerically, which renders the precise determination of the decay exponents difficult. We define the ratio $\alpha = t/L$ and $\alpha_{\max} = \tau_h(L)/L$ and consider equal time correlations for fixed values of α. In the picture of a propagating front that separates a region in the space-time diagram for the chain in which spins are uncorrelated from a region in which they are correlated, one observes quasi-long-range correlations *on the front,* the latter being defined by the ratio $t/L = \alpha_{\max}$. For distances smaller than the distance of maximum correlation or times larger than τ_h the correlations decay slower than exponential in time, e.g., algebraically for bulk spins $[\tilde{C}^{z\psi}(t, r = \text{fixed}) \sim t^{-2}]$. When we vary both space and time with fixed ratio t/L or t/r we get power laws, as long as we stay *behind* the front (i.e., $t \geq \tau_h$). For $\alpha > \alpha_{\max}$ we again observe power laws, but with different exponents; they are listed in Table I.

To conclude, we studied a novel type of dynamically produced long-range correlations in a quantum relaxation process in a quantum spin chain. Starting with a homogeneous initial state the quantum mechanical time evolution according to the Schrödinger equation drives the system into a stationary state, which has algebraically decaying time-dependent autocorrelations but no critical fluctuations. However, *during* the relaxation process spin-spin correlations build up upon arrival of a front of coherent signals, which afterwards decay algebraically in the bulk. *On* the front and behind it for a fixed ratio of space and time scales one observes quasi-long-range order. This does *not* depend on any external parameter like the transverse field. This type of algebraic correlation needs not to be triggered by some tuning parameter and is therefore reminiscent of phenomena in self-organized criticality [12]. The scenario we have reported here is a result of quantum interference and one may expect that a similar one holds for other quantum systems, too. At this point one should mention the possibility of coarsening in quantum systems as reported, for instance, in [13], which is different from the scenario we have reported here.

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