Entanglement Entropy at Infinite-Randomness Fixed Points in Higher Dimensions

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The entanglement entropy of the two-dimensional random transverse Ising model is studied with a numerical implementation of the strong-disorder renormalization group. The asymptotic behavior of the entropy per surface area diverges at, and only at, the quantum phase transition that is governed by an infinite-randomness fixed point. Here we identify a *double-logarithmic* multiplicative correction to the area law for the entanglement entropy. This contrasts with the pure area law valid at the infinite-randomness fixed point in the diluted transverse Ising model in higher dimensions.

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Extensive studies have been devoted recently to understanding ground state entanglement in quantum many-body systems [1]. In particular, the behavior of various entanglement measures at or near quantum phase transitions has been of special interest. One of the widely used entanglement measures is the von Neumann entropy, which quantifies entanglement of a pure quantum state in a bipartite system. Critical ground states in one dimension (1D) are known to have entanglement entropy that diverges logarithmically in the subsystem size with a universal coefficient determined by the central charge of the associated conformal field theory [2]. Away from the critical point, the entanglement entropy saturates to a finite value, which is related to the finite correlation length.

In higher dimensions, the scaling behavior of the entanglement entropy is far less clear. A standard expectation is that noncritical entanglement entropy scales as the area of the boundary between the subsystems, known as the "area law" [3,4]. This area relationship is known to be violated for gapless fermionic systems [5] in which a logarithmic multiplicative correction is found. One might suspect that whether the area law holds or not depends on whether the correlation length is finite or diverges. However, it has turned out that the situation is more complex: numerical findings [6] and a recent analytical study [7] have shown that the area law holds even for critical bosonic systems, despite a divergent correlation length. This indicates that the length scale associated with entanglement may differ from the correlation length. Another ongoing research activity for entanglement in higher spatial dimensions is to understand topological contributions to the entanglement entropy [8].

The nature of quantum phase transitions with quenched randomness is in many systems quite different from the pure case. For instance, in a class of systems the critical behavior is governed by a so-called infinite-randomness fixed point (IRFP), at which the energy scale ϵ and the length scale L are related as $\ln\epsilon \sim L^{\psi}$ with $\psi > 0$. In these systems the off-critical regions are also gapless and the excitation energies in these so-called Griffiths phases scale as $\epsilon \sim L^{-z}$ with a nonuniversal dynamical exponent $z < \infty$. Even so, certain random critical points in 1D are shown to have logarithmic divergences of entanglement entropy with universal coefficients, as in the pure case; these include infinite-randomness fixed points in the random-singlet universality class [9–13] and a class of aperiodic singlet phases [14].

In this Letter we consider the random quantum Ising model in two dimensions (2D), and examine the disorder-averaged entanglement entropy. The critical behavior of this system is governed by an IRFP [15,16] implying that the disorder strength grows without limit as the system is coarse grained in the renormalization group (RG) sense. In our study, the ground state of the system and the entanglement entropy are numerically calculated using a strong-disorder RG method [17,18], which yields asymptotically exact results at an IRFP. To our knowledge this is the first study of entanglement in higher dimensional interacting quantum systems with disorder.

The random transverse Ising model is defined by the Hamiltonian

$$H = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x. \qquad (1)$$

Here the $\{\sigma_i^\alpha\}$ are spin-1/2 Pauli matrices at site i of an $L \times L$ square lattice with periodic boundary conditions. The nearest neighbor bonds $J_{ij}(\geq 0)$ are independent random variables, while the transverse fields $h_i(\geq 0)$ are random or constant. For a given realization of randomness we consider a square block A of linear size ℓ, and calculate the entanglement between A and the rest of the system B, which is quantified by the von Neumann entropy of the reduced density matrix for either subsystems:

$$S = -\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\mathrm{Tr}(\rho_B \log_2 \rho_B). \qquad (2)$$

The basic idea of the strong-disorder RG (SDRG) approach is as follows [17,18]: The ground state of the system is calculated by successively eliminating the largest local terms in the Hamiltonian and by generating a new effective

Hamiltonian in the frame of the perturbation theory. If the strongest bond is J_{ij}, the two spins at i and j are combined into a ferromagnetic cluster with an effective transverse field $\tilde{h}_{(ij)} = \frac{h_i h_j}{J_{ij}}$. If, on the other hand, the largest term is the field h_i, the spin at i is decimated and an effective bond is generated between its neighboring sites, say j and k, with strength $\tilde{J}_{jk} = \frac{J_{ij} J_{ik}}{h_i}$. After decimating all degrees of freedom, we obtain the ground state of the system, consisting of a collection of independent ferromagnetic clusters of various sizes; each cluster of n spins is frozen in an entangled state of the form

$$\frac{1}{\sqrt{2}}(|\underbrace{\uparrow\uparrow\cdots\uparrow}_{n\text{times}}\rangle + |\underbrace{\downarrow\downarrow\cdots\downarrow}_{n\text{times}}\rangle). \quad (3)$$

In this representation, the entanglement entropy of a block is given by the number of clusters that connect sites inside to sites outside the block [Fig. 1]. We note that correlations between remote sites also contribute to the entropy due to long-range effective bonds generated under renormalization.

In 1D the RG calculation can be carried out analytically and the disorder-averaged entropy $\bar{S}_\ell$ of a segment of length ℓ has been obtained as $\bar{S}_\ell = \frac{\ln 2}{6}\log_2 \ell$ [9]. In higher dimensions, the RG method can only be implemented numerically. The major complication in this case is that

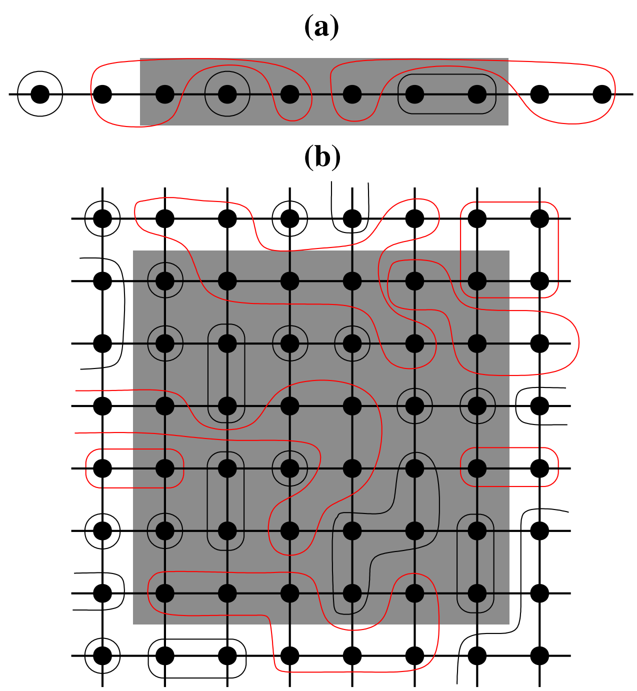

FIG. 1 (color online). An example of the typical ground state in the random quantum Ising model (a) in 1D, and (b) in 2D; it contains a collection of spin clusters of various sizes, which are formed and decimated during the RG. The entanglement of a block (shaded area) is give by the number of decimated clusters (indicated by red loops) that connect the block with the rest of the system.

the model is not self-dual and thus the location of the critical point is not exactly known. To locate the critical point, we can make use of the fact that the excitation energy of the system has the scaling behavior $\ln\epsilon \sim L^{\psi}$ at criticality, while it follows $\epsilon \sim L^{-z}$ in the off-critical regions. In the numerical implementation of the SDRG method, the low-energy excitations of a given sample can be identified with the effective transverse field $\tilde{h}_\infty$ of the last decimated spin cluster, or with the effective coupling $\tilde{J}_\infty$ of the last decimated cluster pair.

In our implementation we set for convenience the transverse fields to be a constant h_0 and the random bond variables were taken from a rectangular distribution centered at $\bar{J} = 1$ with a width $\Delta = 0.5$. The critical point was approached by varying the single control parameter h_0. Although this initial disorder appears to be weak, the renormalized field and bond distributions become extremely broad even on a logarithmic scale [Fig. 2] at the critical point $h_0 = h_c = 1.175$. This indicates the RG flow towards infinite randomness. Slightly away from the critical point, both in the disordered Griffiths phase with $h_0 = 1.18$ and in the ordered Griffiths phase with $h_0 = 1.17$, the distributions have a finite width and obey quantum-Griffiths scaling $h_\infty \sim L^{-z}$. At the critical point one has IRFP scaling $\ln h_\infty \sim L^{\psi}$ and we estimate the scaling exponent as $\psi = 0.55$, quite close to the value $\psi = 0.5$ for the 1D case [17].

Now we consider the entanglement entropy near the infinite-randomness critical point. To obtain the disorder-averaged entanglement entropy $\bar{S}_\ell$ of a square block of size ℓ, we averaged the entropies over blocks in different positions of the whole system for a given disorder realization and then averaged over a few thousand samples. In Fig. 3 we show the entropy per surface unit $\bar{S}_\ell/\ell = \bar{s}_\ell$ for different values of h_0. This average entropy density is found to be saturated outside the critical point, which corresponds to the area law. At the critical point $\bar{s}_\ell$ increases monotonously with ℓ, and the numerical data are consistent with a log-log dependence:

$$\bar{S}_\ell \sim \ell \log_2 \log_2 \ell \quad (4)$$

as illustrated in Fig. 3. In this way we have identified an alternative route to locate the infinite-randomness critical point: it is given by the field h_0 for which the average block entropy at $\ell = L/2$ is maximal. Indeed the numerical results in Fig. 3 predict the same value of h_c as obtained from the scaling of the gaps. We note that the same quantity, the position of the maxima of the average entropy, can be used for the random quantum Ising chain to locate finite-size transition points [19].

The log-log size dependence of the average entropy in Eq. (4) at criticality is completely new; it differs from the scaling behavior observed in 2D pure systems, like the area law, $S_\ell \sim \ell$, for critical bosonic systems [6,7], or a logarithmic multiplicative correction to the area law, $S_\ell \sim \ell \log_2 \ell$, as found in free fermions [5–7,20]. This double-

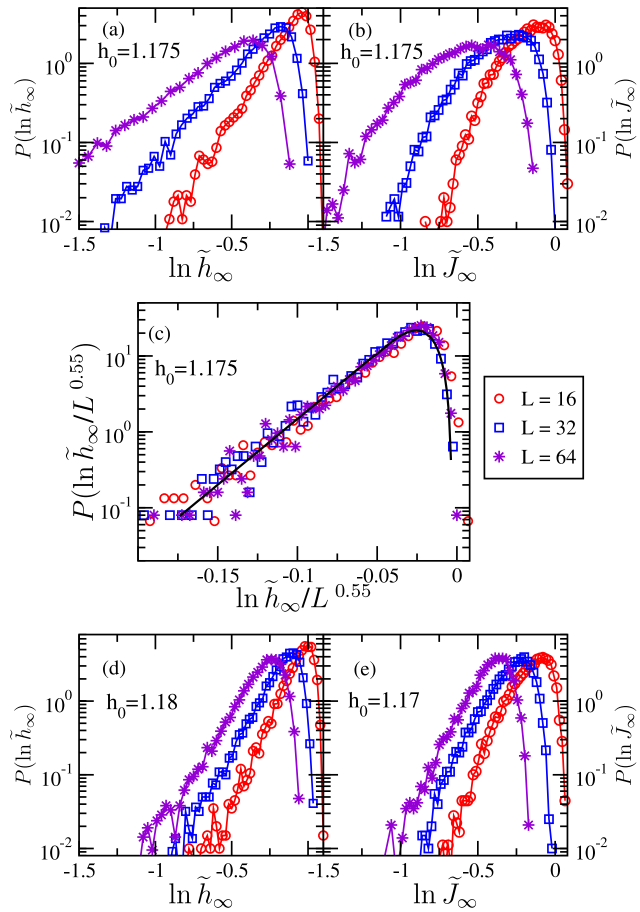

FIG. 2 (color online). The distribution of the last decimated effective log-fields $\ln\tilde{h}_\infty$, and the distribution of the last decimated effective log-bonds $\ln\tilde{J}_\infty$ in the RG calculations. At $h_0 = 1.175$, the distributions, shown in (a) and (b), get broader with increasing system sizes, indicating the RG flow towards infinite randomness; i.e., the system is critical. A scaling plot of the data in (a) using energy-length scaling $\ln\tilde{h}_\infty \sim L^\psi$ with $\psi = 0.55$ is presented in (c). The solid line is just a guide to the eye. The subfigures (d) and (e) show the log-field distribution at $h_0 = 1.18$ and the log-bond distribution at $h_0 = 1.17$, respectively; the distributions show a power-law decaying tail in the low-energy region, which is clear evidence that the system is in the Griffiths phases.

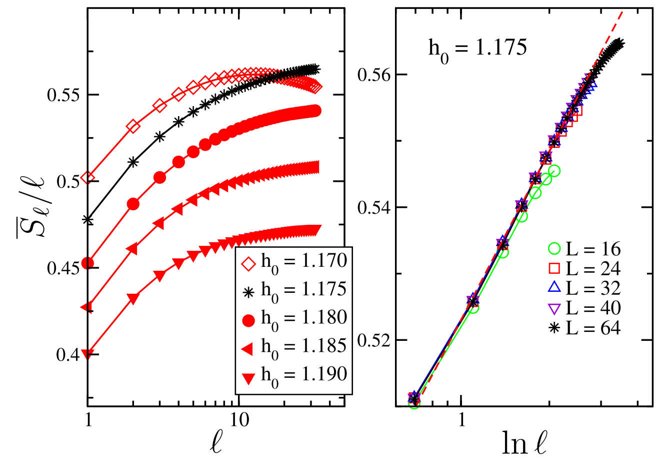

FIG. 3 (color online). Left panel: The disorder-averaged block entropy per surface unit $\bar{S}_\ell/\ell$ vs the linear size of the block ℓ for a system size $L = 64$ for various values of h_0. We observe that the entropy for $\ell = L/2$ reaches its maximum at the critical point $h_c = 1.175$ (cf. Fig. 2). Right panel: The block entropy per surface area vs $\ln\ell$ on a log-scale for different system sizes L at the critical point. The data show a straight line (guided by the dashed line), corresponding to the scaling obeying the area law with a double-logarithmic correction, as given in Eq. (4).

logarithmic correction can be understood via a SDRG argument: In the 1D case a characteristic length scale r at a given RG step is identified with the average length of the effective bonds, i.e., the average size of the effective clusters. At the scale $r(<\ell)$ the fraction of the total number of spins, n_r, that have not been decimated is given by $n_r \sim 1/r$ [17]; these active (i.e., undecimated) spins have a finite probability to form a cluster across the boundary of the block (a segment ℓ in the 1D case) and thus to give contributions to the entanglement entropy. Repeating the renormalization until the scale $r \sim \ell$, the contributions to the entropy are summed up: $\bar{S}_\ell \sim \int_{r_0}^{\ell} dn_r \sim \ln\ell$, leading to the logarithmic dependence of the 1D model [9]. For the 2D case with the same type of RG transformation with a length scale $r < \ell$, the fraction of active spins in the renormalized surface layer of the block is $n_r \sim \ell/r$. Here we have to consider the situation in which some of these active surface spins would form clusters within the surface layer and thus contribute zero entanglement entropy; the number of the active spins that are already engaged in clusters on the surface at RG scale r is proportional to $\ln r$, as known from the 1D case, and only $\mathcal{O}(1)$ of the active surface spins would form clusters connecting the block with the rest of the system. Consequently, the entropy contribution in 2D can be estimated as $\bar{S}_\ell \sim \int_{r_0}^{\ell} dn_r/\ln r \sim \ell \ln\ln\ell$, i.e., a double-logarithmic ℓ dependence, as reflected by the numerical data in Fig. 3.

Based on the SDRG argument described above, the double-logarithmic correction to the area law appears to be applicable for a broad class of critical points in 2D with infinite randomness. For instance, the critical points of quantum Ising spin glasses are believed to belong to the same universality class as ferromagnets since the frustration becomes irrelevant under RG transformation, and the same type of cluster formations as observed in our numerics for the ferromagnet is expected to be generated during the action of the RG. The entanglement entropy at the IRFP is completely determined by the cluster geometries occurring during the SDRG.

Another type of IRFP in higher dimensions occurs in the bond-diluted quantum Ising ferromagnet: The Hamiltonian is again given by (1), but now $J_{ij} = 0$ with probability p and $J_{ij} = J > 0$ with probability $1 - p$. At percolation threshold $p = p_c$ there is a quantum critical line along small nonzero transverse fields, which is controlled by the

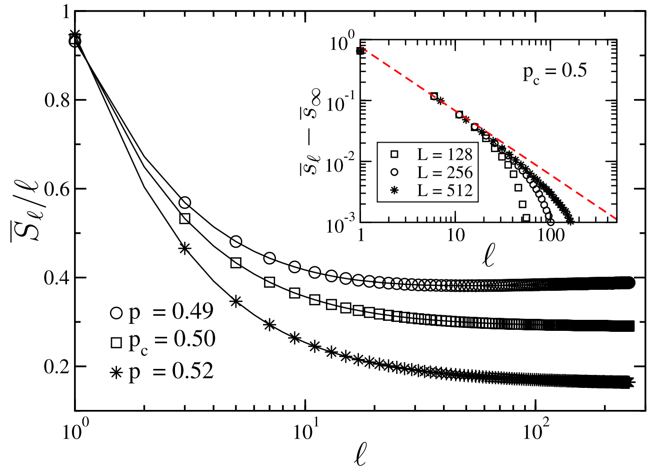

FIG. 4 (color online). The entropy per surface area $\bar{S}_\ell/\ell = \bar{s}_\ell$ vs ℓ near the percolation threshold $p_c = 0.5$ for the 2D bond-diluted Ising model at small transverse fields for $L = 512$. The curves converge to finite values for $\ell \to \infty$, corresponding to the area law. The inset shows $\bar{s}_\ell - \bar{s}_\infty$ as a function of ℓ. $\bar{s}_\infty$ is estimated from $\bar{s}_{L/2}$ at $L = 512$. The dashed line corresponds to ℓ^{-1}.

classical percolation fixed point, and the energy scaling across this transition line obeys $\ln\epsilon \sim L^{\psi}$, implying an IRFP [21]. The ground state of the system is given by a set of ordered clusters in the same geometry as in the classical percolation model—only nearest neighboring sites are combined into a cluster. In this cluster structure, the block entropy, determined by the number of the clusters connecting the block and the rest of the system, is bounded by the area of the block, i.e., $\bar{S} \sim \ell^{d-1}$ with d being the dimensionality of the system. To examine this, we determined the entanglement entropy by analyzing the cluster geometry of the bond-diluted transverse Ising model. Figure 4 shows our results for the square lattice, which follow a pure area law with an additive constant: $\bar{S}_\ell = a\ell + b + \mathcal{O}(1/\ell)$.

To summarize, we have found that the entanglement properties at quantum phase transitions of disordered systems in dimensions larger than 1 can behave quite differently. Generalizing our arguments for the 2D case, we expect for the random bond transverse Ising systems a multiplicative d-fold logarithmic correction to the area law in d dimensions at the critical point, whereas for diluted Ising model at small transverse fields the area law will hold in any dimension $d > 1$ at the percolation threshold. Although both critical points are described by infinite-randomness fixed points, the structure of the strongly coupled clusters in both cases is fundamentally different, reflecting the different degrees of quantum mechanical entanglement in the ground state of the two systems. This behavior appears to be in contrast to one-dimensional systems governed by IRFPs [9].

Other disordered quantum systems in higher dimensions might also display interesting entanglement properties: For instance, the numerical SDRG has also been applied to higher dimensional random Heisenberg antiferromagnets which do not display an IRFP [22]. The ground states involve both singlet spins and clusters with larger moments; therefore, we expect the correction to the area law to be weaker than a multiplicative logarithm and different from the valence bond entanglement entropy in the Néel phase [23].

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