

## Growing Length Scales during Aging in $2d$ Disordered Systems

H. RIEGER, G. SCHEHR and R. PAUL

*Theoretische Physik, Universität des Saarlandes, 66041 Saarbrücken, Germany*

The non-equilibrium dynamics of three paradigmatic models for two-dimensional systems with quenched disorder is studied with a focus on the existence and analysis of a growing length scale during aging at low temperatures: 1) The random bond Ising ferromagnet, 2) the Edwards-Anderson model for a spin glass, 3) the solid-on-solid model on a disordered substrate (equivalent to the sine-Gordon model with random phase shifts). Interestingly, we find in all three models a length scale that grows algebraically with time (up to the system size in cases 1) and 3), up to the finite equilibrium length in case 2)) with a temperature dependent growth exponent. Whereas in cases 1) and 2) this length scale characterizes a coarsening process, it represents in case 3) the growing size of fluctuations during aging.

### §1. Introduction

The non-equilibrium dynamics of disordered, in particular of glassy systems has become a very rich field in recent years and despite many efforts the understanding of non-equilibrium dynamics of disordered and glassy systems in finite dimensions remains a challenging problem. In particular in glasses and spin glasses the aging process displays a very rich phenomenology demanding new theoretical concepts.<sup>1)</sup> But already less complex — and apparently less glassy — systems, like disordered but non-frustrated systems<sup>2)</sup> or even pure systems<sup>3)</sup> reveal interesting and unexpected aging phenomena. One of the most intriguing questions in this context is whether the out-of-equilibrium dynamics is essentially fully determined by a coarsening process (a question that even arises in the more complex spin glass situation<sup>4)</sup>), describable by a growing length scale that characterizes essentially all out-of-equilibrium processes. In this paper we will consider three paradigmatic models for two-dimensional systems with quenched disorder with a focus on existence and analysis of a growing length scale during aging at low temperatures: the random bond Ising ferromagnet, the Edwards-Anderson (EA) model for a spin glass, and the solid-on-solid model on a disordered substrate which is equivalent to the sine-Gordon model with random phase shifts.

### §2. The random bond Ising ferromagnet

As the first example for two-dimensional disordered system we consider the random bond Ising ferromagnet. It is defined by the Hamiltonian

$$H = - \sum_{(ij)} J_{ij} S_i S_j, \quad S_i = \pm 1, \quad (2.1)$$

where the couplings  $J_{ij}$  are non-negative quenched random variables of variance  $\varepsilon$  and the sum is over all nearest neighbor pairs  $(ij)$  on a square lattice of size  $L \times L$  with periodic boundary conditions. This paradigmatic model for a disordered

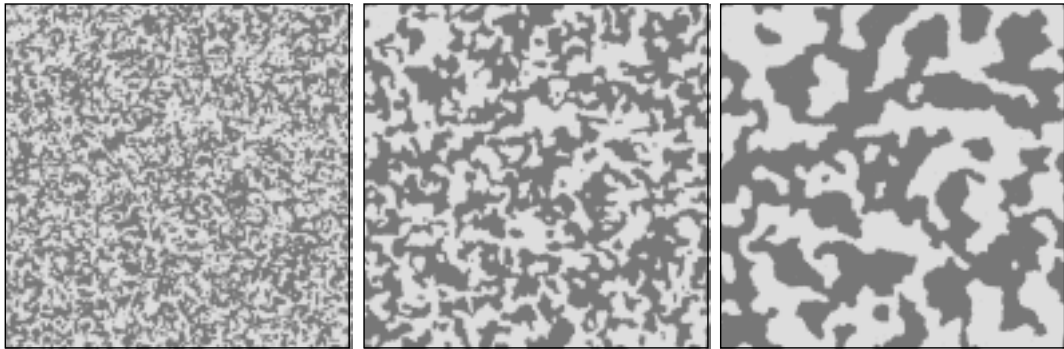


Fig. 1. Domain growth in the RBIM with Glauber kinetics. We show evolution pictures at  $t = 10^2$ ,  $10^4$  and  $10^6$  MCS for a  $512 \times 512$  lattice, after a quench from  $T = \infty$  to  $T = 0.5$  and  $J_{ij}$  uniformly distributed between 0 and 2 ( $J_{ij} \in [0, 2]$ ). The up spins are marked in black, and the down spins are marked in grey.

magnetic system (with bond- or temperature randomness) with an Ising symmetry has a second order phase transition from a paramagnetic to a ferromagnetic phase at a critical temperature  $T_c(\epsilon)$  that decreases with increasing disorder strength  $\epsilon$ . For temperatures  $T$  below  $T_c$  the magnetization  $\overline{\langle m_i \rangle_T}$ , where  $\langle \dots \rangle_T$  means the thermal average and  $\overline{\dots}$  the average over the disorder, takes on a non-vanishing value.

Non-equilibrium dynamics at temperatures below  $T_c$  arises for instance via an instantaneous quench of the systems from the paramagnetic phase to a temperature below  $T_c$ . A stochastic process defined by single spin-flip transition rates defined for instance by the Metropolis rules  $w(S_i \rightarrow -S_i) = 1/(1 + \exp(-\beta(H(S_i) - H(-S_i))))$  models a non-conserved order parameter dynamics and can be studied by computer simulations. For a quench below  $T_c$  the dynamics is a coarsening process during which ferromagnetic domains of a typical lateral extension  $R(t)$  form, where  $t$  is the time elapsed after the quench (see Fig. 1). A standard way to extract this time dependent length scale is via the spatial two-point correlation function  $C(r, t) = \overline{\langle m_i(t)m_{i+r}(t) \rangle_T}$ , which is expected to scale like  $C(r, t) = \tilde{c}(r/R(t))$ .

An important study of the non-conserved random bond Ising model (RBIM) is due to Huse and Henley (HH).<sup>5)</sup> HH argued that coarsening domains are trapped by energy barriers  $E_B(R) \simeq E_0 R^\psi$ , with exponent  $\psi = \chi/(2 - \zeta)$ , where  $\chi$  and  $\zeta$  are the pinning and roughening exponents. For  $d = 2$ , these exponents are known to be  $\chi = 1/3$  and  $\zeta = 2/3$ ,<sup>6)</sup> yielding  $\psi = 1/4$ . As a consequence of the HH scenario one expects the following scaling scenario for the length scale  $R(t)$ :

$$R(t)/R_0 = h(t/t_0) \quad \text{with} \quad h(x) \sim \begin{cases} x^{1/2} & \text{for } x \ll 1, \\ (\ln x)^4 & \text{for } x \gg 1, \end{cases} \quad (2.2)$$

where  $R_0 \sim T^4$  and  $t_0 \sim T^8$ . Instead we find (via an extensive Monte Carlo study, see Ref. 7)) that  $R(t)$  grows algebraically with a temperature and disorder strength dependent exponent  $1/z(T, \epsilon)$ :

$$R(t) \sim t^{1/z(T, \epsilon)} \quad \text{for } t \gg t_0 \quad \text{with} \quad z(T, \epsilon) = 2 + \epsilon/T, \quad (2.3)$$

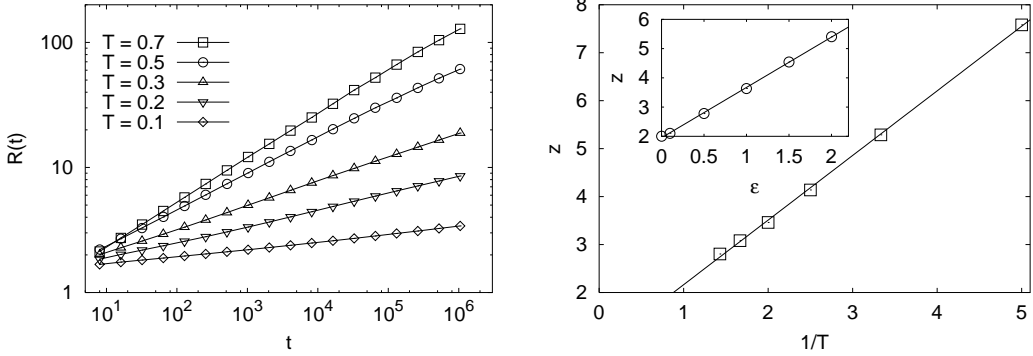


Fig. 2. **Left:** Log-log plot of the correlation length  $R$  vs time  $t$  in the 2d random bond Ising model for different temperatures and  $J_{ij} \in [0, 2]$ . **Right:** Estimates of the exponent  $z$  vs  $1/T$  for the data shown left (note that for  $T > 1 \approx T_c$ , i.e. for  $1/T < 1$  the system is in the paramagnetic phase). Inset shows  $z$  vs the disorder strength  $\epsilon$  for fixed temperature  $T = 0.5$  (the distribution  $P(J)$  is chosen to be uniform over  $[1 - \epsilon/2, 1 + \epsilon/2]$ ). The straight lines (in the main figure and in the inset) represent the analytical prediction  $z = 2 + \epsilon/T$ .

where the time  $t_0$  does not depend on  $T$  and  $\epsilon$  (see Fig. 2). This algebraic growth law with a temperature and disorder strength dependent growth exponent  $\theta$  indicates a logarithmic barrier scaling form  $E_B(R) \sim \epsilon \ln(1 + R)$  in contrast to the algebraic form  $E_B \propto R^\psi$  assumed in the HH picture.

### §3. 2d EA spin glass

Here we consider the two-dimensional Ising spin glass with nearest-neighbor interactions distributed according to a Gaussian with zero mean and variance one

$$H = - \sum_{\langle ij \rangle} J_{ij} S_i S_j \quad \text{with} \quad P(J_{ij}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{J_{ij}^2}{2}\right). \quad (3.1)$$

This model is in a paramagnetic state for all temperature  $T > 0$  but displays a very slow dynamics at low temperatures which can be observed for instance in the non-equilibrium dynamics occurring after a quench from high temperatures. It turns out<sup>8)</sup> that this aging process can be characterized by a coarsening process up to a maximum domain size given by the equilibrium correlation length  $\xi_{\text{eq}}$ .

It is possible to calculate exactly the ground state (GS) of this system using for instance a minimal weight perfect matching algorithm.<sup>9)</sup> Denoting the GS for a particular disorder realization with  $\{S_i^0\}$  we define the local overlap with it as  $q_i^{gs}(t) = S_i(t)S_i^0$ . For a ferromagnetic system (i.e.  $J_{ij} = J > 0$ ) the GS obviously has  $S_i^0 = 1$  and therefore  $q_i$  corresponds to the (time dependent) local magnetization. In Fig. 3 snapshots of the time evolution of the local GS overlap are depicted, showing an increasing average domain size. In contrast to the time evolution of a random bond ferromagnet shown in Fig. 1 even for very large waiting times very small domains exist. These are either very stable clusters because strong bonds have to be broken

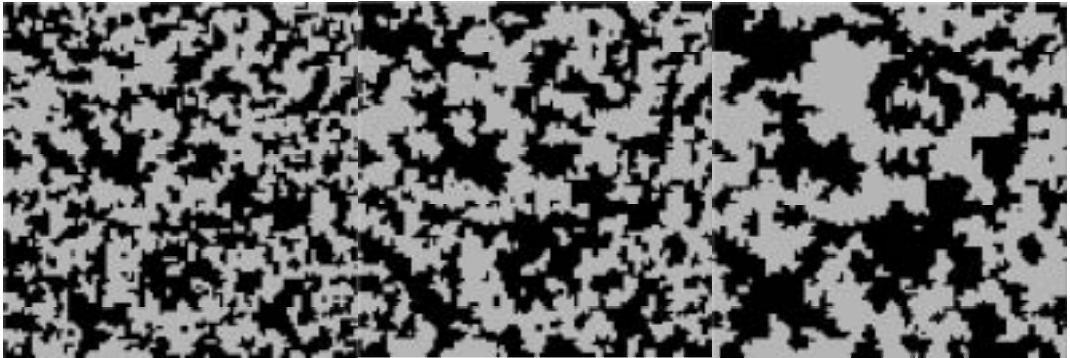


Fig. 3. Domain growth in the  $2d$  EA model with Gaussian couplings for  $T = 0.3$ . The system size is  $L = 100$ , i.e. much smaller than in Fig. 1. The snapshots show the domains relative to the ground state after  $t = 10^2$ ,  $10^4$  and  $10^6$  Monte Carlo sweeps.

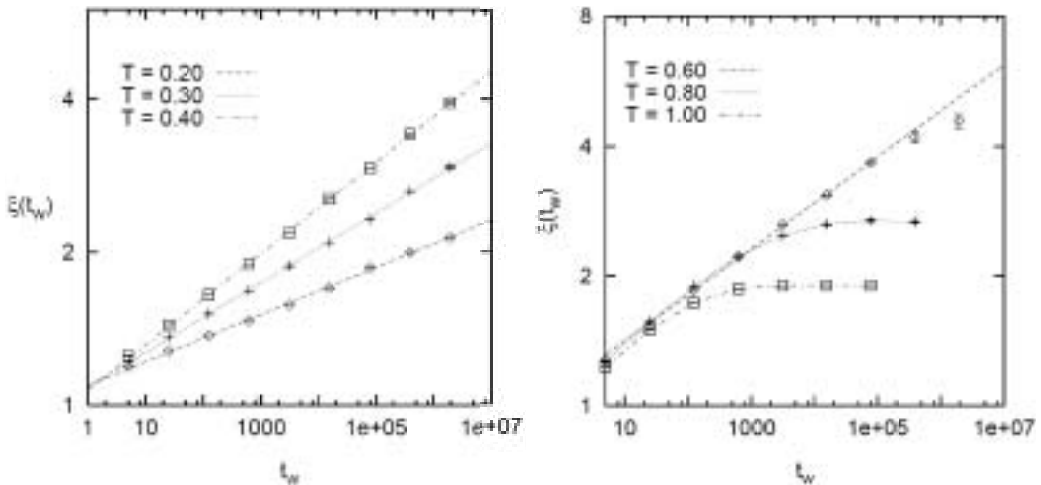


Fig. 4. Correlation length in the  $2d$  EA SG model as a function of time  $t_w$  for different temperatures. **Left:** Fits to an algebraic growth law  $\xi(t_w) \sim t_w^{1/z(T)}$  with  $1/z(T = 0.2) = 0.046$ ,  $1/z(T = 0.3) = 0.068$  and  $1/z(T = 0.4) = 0.090$ . **Right:** For sufficiently large temperatures the time dependent correlation length  $\xi(t)$  saturates at the equilibrium correlation length  $\xi_{\text{eq}}(T)$  within the accessible time window.

to flip the spins or new domains within the bigger ones appear since less strongly bound spins initialize the formation of a new domain.

The spatial correlation function

$$G(r, t) = \overline{\langle q_i^{gs}(t) q_{i+r}^{gs}(t) \rangle} \quad (3.2)$$

allows for a quantitative analysis of the domain size evolution. It turns out that it scales like  $G(r, t) = g(r/\xi(t))$  and we can obtain an estimate for the correlation

length (or typical domain size) via an integral of  $G(r, t)$  over  $r$ . The result is shown in Fig. 4. Note that for increasing temperatures a) the domain growth speeds up, b) the equilibrium correlation length gets smaller. As a consequence of both tendencies one can observe the saturation of the time dependent correlation length at the finite equilibrium correlation length for higher temperatures on the right panel.

We observe that the data for  $\xi(t)$  (in the non-equilibrium regime  $\xi(t) \ll \xi_{\text{eq}}(T)$ ) can very well be fitted by an algebraic growth law with a temperature dependent exponent  $z(T)$ :

$$\xi(t) \propto t^{1/z(T)} \quad \text{with} \quad z(T) \approx (0.23 \cdot T)^{-1} \quad (3.3)$$

which displays again the  $1/T$  behavior that we have encountered already in the last section for the random bond ferromagnet, indicating also here the presence of logarithmic barriers.

#### §4. 2d SOS model on a disordered substrate

Here, we investigate the non equilibrium relaxational dynamics of a solid-on-solid (SOS) model on a disordered substrate, defined on a two dimensional square lattice and described by the following elastic Hamiltonian in terms of height variables  $h_i$

$$H_{\text{SOS}} = \sum_{\langle ij \rangle} (h_i - h_j)^2, \quad h_i \equiv n_i + d_i, \quad (4.1)$$

where  $n_i$  are unbounded discrete variables, i.e.  $n_i \in \{0, \pm 1, \pm 2, \dots\}$  and  $d_i \in [0, 1]$  are uniformly distributed quenched random offsets, uncorrelated from site to site. In the absence of disorder, i.e.  $d_i = 0$ , the model exhibits a roughening transition in the same universality class as the Kosterlitz-Thouless transition,<sup>10)</sup> at a temperature  $T_r$  separating a flat phase at low  $T$  from a logarithmically (thermally) rough one above  $T_r$ . The presence of disorder is known to modify significantly the nature of the transition.<sup>11)</sup> The so-called *superroughening* transition occurs at a temperature  $T_g = T_r/2 = 2/\pi$ . Above  $T_g$ , where the disorder is irrelevant on large length scales, the surface is logarithmically rough again, although below  $T_g$  the system exhibits a glassy phase where the pinning disorder induces a stronger roughness of the interface.

The spatial (2-point) connected correlation function is defined as

$$C(r, t) = \frac{1}{L^2} \sum_i \overline{\langle h_i(t) h_{i+r}(t) \rangle - \langle h_i(t) \rangle \langle h_{i+r}(t) \rangle}, \quad (4.2)$$

which scales as

$$C(r, t) = \mathcal{F}(r/\mathcal{L}(t)) \quad \text{with} \quad \mathcal{L}(t) \sim t^{1/z}. \quad (4.3)$$

Therefore one can estimate  $\mathcal{L}(t)$  by integrating  $C(r, t)$  over  $r$ . In Fig. 5 we show the value of  $\mathcal{L}(t)$  computed in this way for different temperatures. One obtains a rather good fit by a power law  $\mathcal{L}(t) \sim t^{1/z(T)}$ , thus obtaining a value of the  $T$  dependent dynamical exponent. One notices also that  $\mathcal{L}(t)$  approaches an algebraic

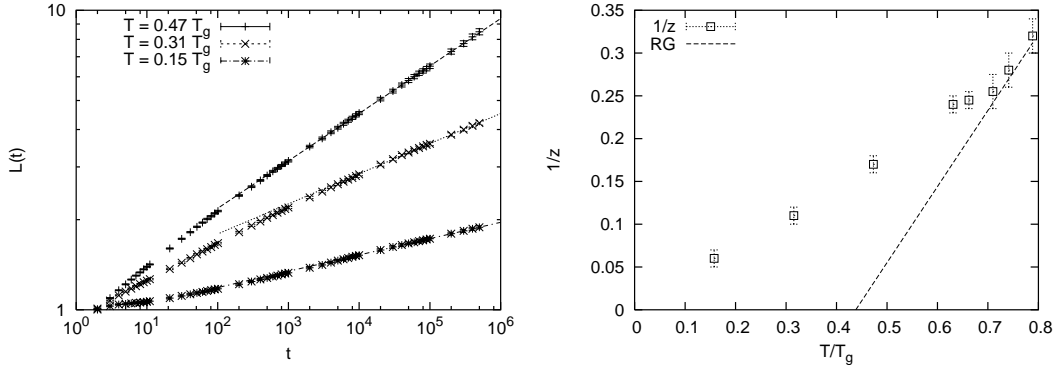


Fig. 5. **Left:** Growing length scale  $\mathcal{L}(t)$  for different temperatures. The solid lines are guides to the eyes. **Right:**  $1/z(T)$  as a function of  $T/T_g$ . The dashed line which shows the result of the one loop RG is drawn without any fitting parameter.

growth after a pre-asymptotic regime which increases with decreasing temperature. Figure 5 shows our estimate for  $1/z(T)$  as a function of  $T$  (for details see Ref. 13)). As expected, the dynamical exponent is a decreasing function of the temperature. One expects that  $z = 2$  for  $T > T_g$  and that it becomes  $T$ -dependent below  $T_g$  with  $z = 2 + 2e^{\gamma_E} \tau + \mathcal{O}(\tau^2)$  as predicted by a one loop RG calculation.<sup>12)</sup> At high temperature  $T > T_g$  and in the vicinity of  $T_g^-$ , it is numerically rather difficult to extract a reliable estimate for the dynamical exponent due to finite size effects. Therefore we restrict ourselves here to lower temperatures  $T < 0.8 T_g$ . For temperature  $T \gtrsim 0.7 T_g$ , the value of  $z$  is still in reasonable agreement with the RG prediction. Around the value  $T^* \simeq 0.63 T_g$ , where  $z \simeq 4$ , the curve  $1/z(T)$  shows an inflection point, below which  $1/z$  decreases linearly with  $T$ . In this regime,  $z(T)$  is well fitted by

$$z(T) \approx 4 \cdot T^*/T \quad \text{for } T \leq T^* \approx 0.63 T_g. \quad (4.4)$$

This behavior  $z \propto 1/T$  is compatible with an activated dynamics over logarithmic barriers, i.e. an Arrhenius type behavior  $t_{\text{typ}} \sim e^{B_{L_{\text{typ}}}/T}$  with  $B_{L_{\text{typ}}} \sim \log L_{\text{typ}}$ .

One would like to relate the length scale  $\mathcal{L}(t)$  to the size of spatially correlated structures like domains or droplets. We first explored the idea that at low temperature, the nonequilibrium dynamics could be understood as a coarsening process reflected in a spatially growing correlation with the ground state (GS). Interestingly, computing the GS of the SOS model on a disordered substrate (4.1) is a minimum cost flow problem for which exists a polynomial algorithm and can therefore be computed exactly.<sup>14)</sup> After determining one GS  $n_i^0$  (note that the GS, which is computed with free boundary conditions, is infinitely degenerated since a global shift of all heights by an arbitrary integer is again a GS), we define for each time  $t$  the height difference  $m_i(t) = n_i(t) - n_i(0)$  and identify the connected clusters (domains) of sites with identical  $m_i(t)$  using a depth-first search algorithm. Notice that for the comparison with the ground state, the Monte Carlo simulations are performed here using free boundary conditions.

In Fig. 6 we show snapshots of these domains for  $T < T_g$ . Starting from a random initial configuration one can for  $T < T_g$  very quickly ( $t \lesssim 100$ ) identify large

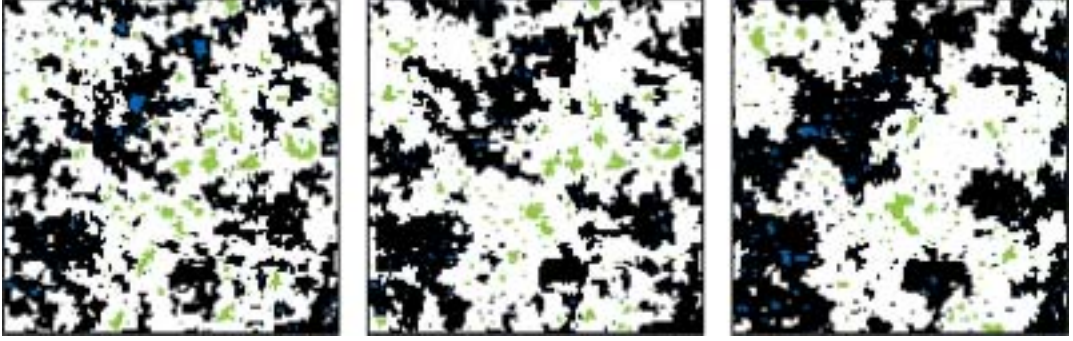


Fig. 6. Snapshot of the height field of the random SOS model relative to the ground state  $m_i(t) = n_i(t) - n_i^0$  for  $T = 0.47T_g$ . The system size is  $L = 128$ . Different colors correspond to different values of  $m_i(t)$ :  $m_i(t) = -2$  in green,  $m_i(t) = -1$  in white,  $m_i(t) = 0$  in black and  $m_i(t) = +1$  in blue and so on. Note that large domains in white and black persist and change only slowly in time.

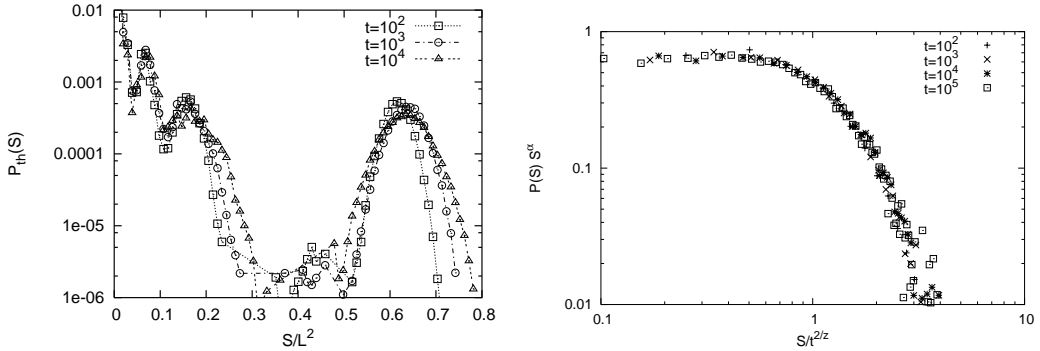


Fig. 7. **Left:** Size distribution  $P_{\text{th}}(S, t)$  (see definition in the text) for different times  $t$ . Here  $T = 0.47T_g$ . **Right:**  $S^\alpha P_{\text{droplet}}^{\text{GS}}(S, t)$  with  $\alpha = 1.9 \pm 0.1$  as a function of  $S/t^{2/z}$  with  $2/z = 0.26 \pm 0.03$ . Here the initial condition is the ground state and  $T = 0.3T_g$ .

domains that evolve only very slowly at later times. On the other hand for  $T > T_g$  the configurations decorrelate very quickly in time. To make this analysis more quantitative, we determined the cluster size distribution  $P_{\text{th}}(S, t)$  for one realization of the disorder (and for different realizations of the thermal noise).

As shown on Fig. 7,  $P_{\text{th}}(S, t)$  starts to develop a peak at a rather large value  $S^*(t)$  on the earlier stage of the dynamics (this peak also develops if we start with a random initial configuration). It turns out that  $S^*(t)$  is the size of the largest connected flat cluster of the ground state configuration  $n_i^0 = C^{\text{st}}$ . On the time scales presented here, as time  $t$  is growing, this peak remains stable  $S^*(t) \simeq C^{\text{st}}$ , implying that the system is *not* coarsening. At later times, as suggested by simulations on smaller systems, this peak progressively disappears and the distribution becomes very flat. We also checked that the mean size of these connected clusters is not directly related to  $\mathcal{L}(t)$ . One has however to keep in mind that we are computing the *connected* correlation functions, i.e. we measure the thermal fluctuations of the height profile around its mean (typical) value  $\langle h_i(t) \rangle$ . Therefore, we believe that

these connected correlations are instead related to the broadening of this “stable” peak (Fig. 7), i.e. the fluctuations around this typical state at time  $t$ .

To characterize more precisely the fluctuations around this cluster, we identify “droplets” by initializing the system in the ground state itself  $n_i(t=0) = n_i^0$ . At low temperature, and on the time scales explored here, one expects that the ground state represents a good approximation of a typical configuration, i.e.  $\langle n_i(t) \rangle \simeq n_i^0$ . We compute the distribution  $P_{\text{droplet}}^{\text{GS}}(S, t)$  of the sizes of the connected clusters with a common value of  $m_i(t) \neq 0$ . It turns out, as shown in Fig. 7, that  $P_{\text{droplet}}^{\text{GS}}(S, t)$  obeys the scaling form

$$P_{\text{droplet}}^{\text{GS}}(S, t) = \frac{1}{S^\alpha} \mathcal{F}_{\text{droplet}}^{\text{GS}}\left(\frac{S}{\mathcal{L}^2(t)}\right), \quad \alpha = 1.9 \pm 0.1, \quad (4.5)$$

where  $\alpha$  is independent of  $T$  within the accuracy of our data and  $\mathcal{L}(t) \sim t^{1/z}$ . The value of  $z$  in Eq. (4.5) is in good agreement with the one extracted from the 2-point correlation function  $C(r, t) = \mathcal{F}(r/\mathcal{L}(t))$  [Eq. (4.3)].

## §5. Conclusion

We studied the time dependent correlation length  $R(t)$  in three models of disordered systems in two dimensions that are characterized by distinct features: 1) The random bond Ising ferromagnet as an example for a random system that has long range order at low temperatures and is expected to perform a simple (but slow) coarsening process after a temperature quench into the ordered phase. 2) The Edwards-Anderson spin glass model as an example for a frustrated system without a finite temperature critical point and an ordered phase but with an extremely slow dynamics and a large correlation length at very low temperatures. 3) The disordered SOS model as a model with a critical point and a low temperature phase without long range order but infinite correlation length. Surprisingly, in spite of the pronounced differences between these systems we find that all three show an algebraic dependence of the correlation length on the age  $t$  of the system  $R(t) \sim t^{1/z(T)}$  and that the exponent  $z$  (which would be identical to the dynamical exponent if the system is critical) depends linearly on the inverse temperature:

$$z(T) \propto 1/T. \quad (5.1)$$

If the dynamics at low temperatures in all three systems is thermally activated, this behavior hints at a logarithmic scaling behavior of the energy barriers as a function of their size. In the disordered SOS model one would actually expect such a scaling.<sup>13)</sup> For the EA spin glass (in  $2d$ ) the situation is complicated by the fact that the ground state is not expected to be stable with respect to thermal fluctuations, i.e. in principle excitations of increasing size would cost less and less energy — therefore a logarithmic barrier scaling comes a bit as a surprise. Finally for the random bond Ising model a simple scaling picture<sup>5)</sup> based on the scaling behavior of the domain walls in this model would predict an algebraic energy scaling — resulting in a formally infinite value for  $z$ , which is not confirmed by our results.<sup>7)</sup> Hence we have to conclude



with the observation that the common behavior (5.1) of the growth exponent  $z$  in 2d disordered models indicates a more complicated and yet hidden mechanism that is active in the non-equilibrium dynamics of these systems at low temperatures at least during the first 10 decades of the aging process.

We also would like to emphasize the fact that the physical interpretation of the growing length scale in the three systems under consideration in this paper is quite different: In the random bond ferromagnet it is simply the typical transverse domain size, where domains are easily identified as connected clusters of common magnetization sign. In the EA spin glass model the length scale is also determined by a domain size — where domains are defined as connected clusters of spins with common orientation with respect to one of the two ground states. These domains grow steadily up to a maximum size set by the equilibrium correlation length. In the disordered SOS model, however, the growing length scale is *not* connected to growing domains — actually the system settles quite fast after the temperature quench into a configuration that has a pretty large overlap with one of the ground states. Instead of growing further these initially very large domains thermal fluctuations of increasing size destroy these domains — and it is the spatial extent of these fluctuations that is characterized by the growing length scale studied here.

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