

Optimal number of agents in a collective search and when to launch themHugues Meyer^{1,2} and Heiko Rieger²¹*School of Physics and Astronomy, University of Nottingham, Nottingham NG7 2RD, United Kingdom*²*Department of Theoretical Physics and Center for Biophysics, Saarland University, 66123 Saarbrücken, Germany*

(Received 11 January 2024; revised 15 May 2025; accepted 20 May 2025; published 6 June 2025)

Search processes often involve multiple agents that collectively search a randomly located target. While increasing the number of agents usually decreases the time at which the first agent finds the target, it also requires resources to create and sustain more agents. In this paper, we raise the question of the optimal timing for launching multiple agents in a search in order to reach the best compromise between minimizing the overall search time and minimizing the costs associated with launching and sustaining agents. After introducing a general formalism for independent agents in which we allow them to be launched at arbitrary times, we investigate by means of analytical calculations and numerical optimization the optimal launch strategies to optimize the quantiles of the search cost and its mean. Finally, we compare our results with the case of stochastic resetting and study the conditions under which it is preferable to launch new searchers rather than resetting the first one to its initial position.

DOI: [10.1103/PhysRevE.111.064112](https://doi.org/10.1103/PhysRevE.111.064112)**I. INTRODUCTION**

The term “search processes” encompasses any phenomenon in which the encounter of agents with a target is important [1]. They are encountered in a wide variety of systems across many length and time scales. This includes chemical reaction kinetics [2,3], biological processes at the molecular scale such as macromolecules searching for reactive sites to trigger biochemical reactions [4], or at the cellular level in the immune search for pathogens [5–7], ecological phenomena such as animal foraging and hunting [8–12], or even searches by artificial agents such as robot swarms used in rescue missions [13,14].

Although stochastic search processes differ in nature, they share common features that can be treated equally using the tools of statistical physics. In particular, they share the need of being optimized in a certain way, raising the following question: How should one tune the free parameters of the process in order to make the search as efficient as possible? In a very large majority of studies, the search efficiency is fully defined in terms of the search time, i.e., the time required by the searchers to find and catch the target. Typically, the inverse of the mean first-passage time is used as an estimator of the search efficiency, which one attempts to maximize. For individual agents evolving according to stochastic dynamics, a lot of work has been dedicated over the past years to identify search strategies, meaning parameter sets of the stochastic search process, that minimize the mean first-passage time (MFPT). Various classes of processes have been investigated, among which are intermittent or Lévy walks [15–20], stochastic resetting [21–25], and non-Markovian searches [26–30].

However, it is only rarely that the search efficiency has been defined to account for other quantities than the search time, such as the energy deployed for the search to succeed or the amount of external resources required.

More recently, some effort has been put into understanding and quantifying collective searches, i.e., identifying optimal strategies that multiple searchers should follow when they are looking for a common target [31–35]. Most studies focused on the MFPT of independent searchers [36–38], but also a few instances for interacting searchers have been studied [32,39–41]. In particular, similarly to individual searches, the search efficiency is almost exclusively defined inversely proportional to the overall search time. In addition, the number of searchers has rarely been considered as a parameter to optimize. However, while it is clear that for nearly all search processes the mean search time decreases with the number of searchers, adding more searchers may have a non-negligible cost in terms of the required resources. For a human search problem where one needs to pay agents and material resources to perform the search, one can very easily understand that it is not optimal to hire as many agents as possibly available, as it would have a huge financial cost. Similarly, in an immune response process, hiring more cells to find a pathogen requires a substantial amount of metabolic energy. This is precisely the motivation for the question that we are raising in this paper: Given the cost associated with launching and sustaining an agent in a collective search process, what is the optimal number of them and when should they be launched?

To address this general question, the paper is organized as follows. In Sec. II we formalize the problem and discuss the assumption on which our work relies. We also introduce a definition of the search cost and discuss its various contributions. Section III is dedicated to the mean search cost. We first treat the two-searcher problem to gain an intuition of the problem, and show that a unique optimum exists if the single-agent survival probability (SASP) and its derivative

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are logarithmically convex. We then derive general analytic results on the optimal number of searchers to launch initially and later perform numerical optimization on a selection of test cases that combine relevant short- and long-time behavior of the SASP to identify the details of the optimal launch strategies. Section IV deals with the minimization of the quantiles of the search cost, i.e., the probability that the cost does not reach too large values. We show that the whole problem can be solved for logarithmically convex SASPs and we therefore derive the optimal strategies that minimize the quantiles of the search cost. We then apply our result to a specific example, namely, an exponentially decaying SASP. Section V presents results of numerical simulations on an example of the search process and compares the data with the predictions made in the previous sections. Finally, in Sec. VI we draw a comparison between our formalism and search processes subject to stochastic resetting, which have attracted a lot of attention in recent years. We show with the canonical example of the one-dimensional diffusive search under which conditions launching new searchers is preferable to resetting the first searcher to its initial position. We conclude in Sec. VII, in which we discuss possible adaptations of our work to more complex problems.

II. GENERAL FORMALISM

A. Target survival probability

The general problem that we consider is the following. A target is initially located at position \mathbf{r}_T in a search domain that can be finite or infinite. We have at our disposal a supply of N agents that are initially out of the search domain and ready to be launched into the search. Once in the search domain, each agent moves within it according to a certain stochastic dynamics. The i th agent is launched at time $t_i \geq t_{i-1}$, referred to as the *launch time*, and position \mathbf{r}_i . We set $t_1 = 0$ and we note $\Delta_i = t_i - t_{i-1}$ the i th *launch interval*. The overall process terminates once one searcher finds the target. A sketch of the setup is depicted in Fig. 1. Depending on the search process, *finding* the target can be achieved whenever an agent comes in the vicinity of the target for the first time, while it may take several attempts for so-called reactive targets which need to interact with the agents in a certain way in order to trigger the initiation of a subsequent process, e.g., in chemical reactions. This distinction will be irrelevant in what follows and both scenarios can be treated by our formalism.

The individual processes can be arbitrary, provided that the SASP of the target, $s_i(T, \mathbf{r}_i, \mathbf{r}_T)$, associated with the searcher i , i.e., the probability that the searcher i has not found the target until time t , is well defined. Note that s_i could be different for each searcher if their dynamics is not identical. Because the searchers are independent, the overall survival probability of the target is given by

$$S(T, \{t_i, \mathbf{r}_i\}, \mathbf{r}_T) = \prod_{k=1}^n s_k(T - t_k, \mathbf{r}_k, \mathbf{r}_T) \quad (1)$$

for $t_n \leq T < t_{n+1}$. Defining $\varrho_T(\mathbf{r}_T)$ and $\varrho_S(\mathbf{r}_1, \dots, \mathbf{r}_N)$ as the probability distributions for the target position \mathbf{r}_T and initial coordinates of each searcher $\mathbf{r}_1, \dots, \mathbf{r}_N$, respectively, the

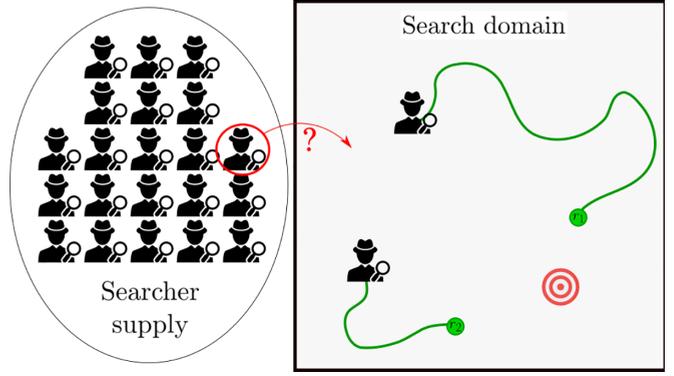


FIG. 1. Sketch of the problem under study. A target is to be found in a search domain and we have a supply of searching agents at our disposal. Progressively, agents are launched into the search domain until the target has been found by one of them. The main question of the paper is then, given the probability for an individual agent to find the target at a certain time after being launched, how should one optimize the timing of the launches in order to minimize the total cost of the process?

averaged survival probability is defined as

$$\bar{S}(T, \{t_i\}) = \int d\mathbf{r}_T \int d\mathbf{r}_1 \cdots \int d\mathbf{r}_N \varrho_T(\mathbf{r}_T) \varrho_S(\{\mathbf{r}_i\}) \times S(T, \{t_i, \mathbf{r}_i\}, \mathbf{r}_T). \quad (2)$$

In order to proceed we now assume that (i) the single-agent survival probabilities are identical, i.e., $s_k = s$, and (ii) searchers' launch positions are independent and identically distributed, i.e., $\varrho_S(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{k=1}^N \rho_s(\mathbf{r}_k)$. With this we find

$$\bar{S}(T, \{t_i\}) = \int d\mathbf{r}_T \varrho_T(\mathbf{r}_T) \left(\prod_{k=1}^n \int d\mathbf{r}_k \rho_s(\mathbf{r}_k) s(T - t_k, \mathbf{r}_k, \mathbf{r}_T) \right) \quad (3)$$

again for $t_n \leq T < t_{n+1}$. From here, a number of search processes are such that Eq. (3) can be simplified into the form

$$\bar{S}(T, \{t_i\}) = \prod_{k=1}^n \bar{s}(T - t_k). \quad (4)$$

This includes two main classes of processes. First is for a target fixed at a deterministic position \mathbf{r}_T^0 , i.e., $\varrho_T(\mathbf{r}_T) = \delta(\mathbf{r}_T - \mathbf{r}_T^0)$. In this case we have $\bar{s}(T - t_k) = \int d\mathbf{r}_k \rho_s(\mathbf{r}_k) s(T - t_k, \mathbf{r}_k, \mathbf{r}_T^0)$. Second is a finite translationally invariant search domain \mathcal{V} with periodic boundary conditions such that \bar{s} depends on \mathbf{r}_k and \mathbf{r}_T through $\mathbf{r}_k - \mathbf{r}_T$, and where agents are launched homogeneously in the domain, i.e., $\rho_s(\mathbf{r}_k) = 1/V$ where V is the volume of the search domain. Then we obtain

$$\begin{aligned} & \int_{\mathcal{V}} d\mathbf{r}_k \rho_s(\mathbf{r}_k) s(T - t_k, \mathbf{r}_k - \mathbf{r}_T) \\ &= \frac{1}{V} \int_{\mathcal{V} + \mathbf{r}_T} d\mathbf{r}_k s(T - t_k, \mathbf{r}_k) \\ &= \frac{1}{V} \int_{\mathcal{V}} d\mathbf{r}_k s(T - t_k, \mathbf{r}_k) \\ &= \bar{s}(T - t_k), \end{aligned} \quad (5)$$

where the second equality comes from periodicity. Note that these two examples are not necessarily the only possible ones that would result in Eq. (4).

From this point on, we will focus on situations where the joint probability $\bar{S}(t)$ can be written in the form of Eq. (4) such that our entire analysis will be made in terms of the averaged single-agent survival probability $\bar{s}(t)$. This will be the central quantity of the paper. Note, however, that this object can still depend on the initial distance between searchers and the target if their respective initial probabilities are δ distributed. We finally make two assumptions on $\bar{s}(t)$: (i) $\lim_{t \rightarrow \infty} \bar{s}(t) = 0$, i.e., the probability for one agent to find the target eventually is 1, and (ii) the function $\bar{s} : \mathbb{R}^+ \rightarrow [0, 1]$ is bijective such that we can define an inverse function \bar{s}^{-1} which obeys $\bar{s}^{-1}(\bar{s}(t)) = t$ for all $t > 0$.

Finally, we emphasize that the launch times t_i are chosen to be deterministic; i.e., they are not drawn from a certain probability distribution and do not require to be averaged over. This choice is motivated by the fact that search processes are often optimized when control parameters are not random. This is, e.g., well known for stochastic resetting where the resetting times are deterministic in optimal strategies [22], or in random search processes with memory where optimal transition rates are also found to be deterministic [27].

B. Search cost

In order to identify optimal launch strategies, we first need to define a search cost, i.e., an objective function of the launch times $\{t_i\}$ that we will aim at minimizing. As mentioned in the Introduction, we want to account not only for the overall search time but also for the resources required to launch each searcher and to sustain it. We thus aim at constructing a cost function that depends on the total number \mathcal{N} of searchers launched before the target is found and the total time \mathcal{T} spent by all searchers in the search. These quantities can be defined in terms of the total search time T , reading

$$\mathcal{N} = \sum_{i=1}^N \Theta(T - t_i), \quad (6)$$

$$\mathcal{T} = \sum_{i=1}^N (T - t_i) \Theta(T - t_i), \quad (7)$$

where Θ is the Heaviside step function. A relevant search cost K should be a function of T , \mathcal{N} , and \mathcal{T} , where each of them should contribute independently. The simplest form that can be constructed is thus a weighted sum of three contributions, namely,

$$K = J_T T + J_S \mathcal{T} + K_L \mathcal{N}. \quad (8)$$

The first term, weighted by the *target cost rate* J_T , quantifies a cost associated with the presence of the target and can be interpreted as a rate of damage due to the presence of the target. The second term, weighted by the *searcher sustaining rate* J_S , quantifies the amount of resources required to sustain one searcher per unit time. Finally, the last term, weighted by the *searcher launch cost* K_L , quantifies the amount of resources required to launch a searcher. The values of J_T , J_S , and K_L are constant for each specific search process and reflect the relative importance of each contribution to the search cost. Of

course, more complicated definitions of the search cost can be chosen where the dependence on T , \mathcal{N} , and \mathcal{T} are nonlinear, but this would be beyond the scope of this paper and is briefly discussed in the Conclusion. For compactness, we introduce the normalized parameters $\gamma = J_S/J_T$ and $\kappa = K_L/J_T$ and set $J_T = 1$ as our cost rate unit for the rest of the paper.

Note that on the level of Eq. (8), K is a stochastic quantity as T , \mathcal{N} , and \mathcal{T} differ from one trajectory to the other. Optimizing a search strategy implies to minimize a statistical estimate of K with respect to the launch times t_i . In what follows, we will first focus on the optimization of the mean search cost, and then on the quantiles of the search cost. In both cases, we will show that some general results can be derived if one assumes the SASP to be logarithmically convex, which is actually observed in many search processes.

III. OPTIMIZING THE MEAN SEARCH COST

We start by analyzing the mean search cost, the most common statistical estimate of the search cost. Given the joint first-passage time distribution $\bar{R}(t) = -\bar{S}'(t)$, the mean first-passage time \bar{T} is classically obtained as $\bar{T} = \int_0^\infty t \bar{R}(t) dt = \int_0^\infty \bar{S}(t) dt$ using integration by parts. Similarly, we have

$$\begin{aligned} \bar{\mathcal{T}} &= \sum_{n=1}^{\infty} \int_0^\infty \frac{d}{dt} ((t - t_n) \Theta(t - t_n)) \bar{S}(t) dt \\ &= \sum_{n=1}^{\infty} n \int_{t_n}^{t_{n+1}} \bar{S}(t) dt, \end{aligned} \quad (9)$$

and finally

$$\bar{\mathcal{N}} = \sum_{n=1}^{\infty} \int_0^\infty \frac{d\Theta(t - t_n)}{dt} \bar{S}(t) dt = \sum_{n=1}^{\infty} \bar{S}(t_n). \quad (10)$$

Combining the contributions, we obtain the mean search cost as

$$\bar{K}(\{t_i\}) = \sum_{n=1}^N \left[\kappa \bar{S}(t_n) + (1 + n\gamma) \int_{t_n}^{t_{n+1}} \bar{S}(t) dt \right]. \quad (11)$$

Minimizing this quantity in general for an arbitrary SASP is not possible. However, we can still derive interesting results if we assume that there exists only one local minimum of \bar{K} . In this section, we will first take a detour through the case $N = 2$ in order to get an intuition for the conditions for such a unique minimum to exist. We will then derive general results under this assumption, and finally perform numerical optimization for certain test cases.

A. Two searchers

For two searchers, the mean search cost is a function of only one variable, namely, the launch time t_2 of the second searcher. Although we do not aim at determining analytically the optimal value for t_2 in general, we want to understand the conditions for which a unique optimal strategy can be found.

The mean search cost \bar{K} of the two-searcher process reads as follows:

$$\begin{aligned} \bar{K}(t_2) &= \kappa(1 + \bar{s}(t_2)) + (1 + \gamma) \int_0^{t_2} \bar{s}(t) dt \\ &\quad + (1 + 2\gamma) \int_0^\infty \bar{s}(t) \bar{s}(t + t_2) dt. \end{aligned} \quad (12)$$

Upon integration by parts, the first derivative of \bar{K} can be written as

$$\bar{K}'(t_2) = \kappa \bar{s}'(t_2) - \gamma \bar{s}(t_2) - (1 + 2\gamma) \int_0^\infty \bar{s}'(t) \bar{s}(t + t_2) dt. \quad (13)$$

The extrema of \bar{K} are found such that $\bar{K}' = 0$. For reasons that will become apparent later, we rewrite this condition as

$$\kappa = \frac{\gamma \bar{s}(t_2) + (1 + 2\gamma) \int_0^\infty \bar{s}'(t) \bar{s}(t + t_2) dt}{\bar{s}'(t_2)}. \quad (14)$$

Differentiating Eq. (13) with respect to t_2 and substituting κ with the right-hand side of Eq. (14) we can write $\bar{K}''(t_2)$ as

$$\bar{K}''(t_2) = -\bar{s}'(t_2) \frac{d}{dt_2} \left[\frac{g(t_2)}{\bar{s}'(t_2)} + \gamma \frac{\bar{s}(t_2) + g(t_2)}{\bar{s}'(t_2)} \right], \quad (15)$$

where we have defined

$$g(t_2) = \int_0^\infty \bar{s}'(t) \bar{s}(t + t_2) dt. \quad (16)$$

First, we have

$$\frac{d}{dt_2} \left(\frac{g}{\bar{s}'} \right) = \int_0^\infty \bar{s}'(t) \frac{\bar{s}'(t + t_2) \bar{s}'(t_2) - \bar{s}''(t_2) \bar{s}(t + t_2)}{\bar{s}'(t_2)^2} dt. \quad (17)$$

If \bar{s} is logarithmically convex, then for any $x, y \in \mathbb{R}^+$ with $x < y$ it holds that $\bar{s}''(x) \bar{s}(y) > \bar{s}'(x) \bar{s}'(y)$ such that the right-hand side of Eq. (17) is positive.

Then, we can write upon integration by parts

$$\frac{\bar{s}(t_2) + g(t_2)}{\bar{s}'(t_2)} = 2 \int_0^\infty \frac{\bar{s}(t + t_2) - \bar{s}(t) \bar{s}(t_2)}{\bar{s}(t_2)} \frac{\bar{s}'(t + t_2)}{\bar{s}'(t_2)} dt. \quad (18)$$

Now, note that

$$\begin{aligned} \frac{d}{dt_2} \frac{\bar{s}(t + t_2) - \bar{s}(t) \bar{s}(t_2)}{\bar{s}'(t_2)} \\ = \frac{\bar{s}'(t + t_2) \bar{s}(t_2) - \bar{s}(t + t_2) \bar{s}'(t_2)}{\bar{s}'(t_2)^2} \end{aligned} \quad (19)$$

and

$$\frac{d}{dt_2} \frac{\bar{s}'(t + t_2)}{\bar{s}'(t_2)} = \frac{\bar{s}''(t + t_2) \bar{s}'(t_2) - \bar{s}'(t + t_2) \bar{s}''(t_2)}{\bar{s}'(t_2)^2}. \quad (20)$$

Assuming logarithmic convexity for both \bar{s} and $-\bar{s}'$, both these terms are positive, which implies that the integrand in h is the product of two positive, increasing functions of t_2 , which makes h also a positive, increasing function of t_2 . This proves that the derivative of $(\bar{s} + g)/\bar{s}'$ is positive.

We therefore conclude that if $\bar{s}(t)$ and $-\bar{s}'(t)$ are log-convex then \bar{K}'' is the sum of two positive terms; i.e., $\bar{K}'' > 0$ for any local extremum of \bar{K} . Any local extremum is therefore

a local minimum. Now, suppose that there are two local minima. By continuity and differentiability, there must also be a local maximum between both of them, which is impossible. We conclude that \bar{K} has at most one local minimum and no local maximum. However, we cannot determine the value of t_2 that minimizes \bar{K} in general as Eq. (13) cannot be inverted for an arbitrary SASP $\bar{s}(t)$. Some simple practical cases can, however, be treated analytically. For the sake of example, let us consider an exponentially decaying SASP of the form $\bar{s}(t) = e^{-\lambda t}$, where λ^{-1} is the single-agent mean first-passage time. There, we have

$$\bar{K}(t_2) = \left(\kappa - \frac{1}{2\lambda} \right) e^{-\lambda t_2} + \kappa + (1 + \gamma) \lambda^{-1}. \quad (21)$$

If $\kappa \lambda > 1/2$, then $\bar{K}(t_2)$ decreases monotonically with t_2 such that the optimal search cost is reached for $t_2 \rightarrow \infty$; i.e., one should never launch a second searcher. However, if $\kappa \lambda < 1/2$ the cost is minimized for $t_2 = 0$; i.e., one should launch the searcher together with the first one.

For an arbitrary number of searchers, N , it is much more difficult to prove that the assumption of logarithmic convexity on $\bar{s}(t)$ is sufficient for the uniqueness of a local minimum of the mean search cost. However, in the next sections we will show numerically that this condition still holds for $N > 2$.

B. Conditions for simultaneous launching

Although we cannot analytically predict the details of optimal launch procedures for arbitrary SASPs, we can derive a criterion to decide whether it is beneficial to launch searchers simultaneously or not in order to minimize the mean search cost. Before we proceed, let us first introduce some notations:

$$s_k = \bar{s}(t_k), \quad (22)$$

$$I_{nk} = \int_{t_n}^{t_{n+1}} \bar{s}(t) \zeta(t - t_k) dt, \quad (23)$$

$$r_{nk} = s_k \zeta(t_n - t_k). \quad (24)$$

With these notations the gradient of \bar{K} reads

$$\nabla_k \bar{K} = -\gamma s_k - \sum_{n=k}^N (1 + n\gamma) I_{nk} + \kappa \left(\sum_{n=1}^{k-1} r_{kn} - \sum_{n=k+1}^N r_{nk} \right), \quad (25)$$

where $\nabla_k = \partial_{t_k}$. We also note $\mathbf{t}_N = \{t_2, \dots, t_N\}$.

Let us first assume that the mean search cost $\bar{K}(\mathbf{t}_N)$ has at most one unique local minimum located at a point \mathbf{t}_N^* . At this location, two possible configurations can in principle be observed for the launch times i . Either t_i^* is strictly larger than t_{i-1}^* and the gradient $\nabla_i \bar{K}_N(\mathbf{t}_N^*)$ vanishes, i.e., it is an actual local extremum of \bar{K} with respect to t_i , or $t_i^* = t_{i-1}^*$, i.e., the i th and $(i-1)$ th searchers are launched simultaneously, which happens only if the derivative of \bar{K} with respect to t_i at this location is positive. Let us now proceed by contradiction.

Assume that \bar{K} is minimized at a point \mathbf{t}_N^* , where $t_{k+1}^* = t_k^*$ and $t_k^* > t_{k-1}^*$ for a certain $k > 2$. Following our previous observation, this implies that $\nabla_{k+1} \bar{K} > 0$ and $\nabla_k \bar{K} = 0$. In addition, for $t_{k+1} = t_k$, we have $s_k = s_{k+1}$, $I_{n,k} = I_{n,k+1}$, $r_{k,n} = r_{k+1,n}$, $r_{n,k} = r_{n,k+1}$, and $I_{k,k} = 0$. When computing the

difference between $\nabla_k \bar{K}$ and $\nabla_{k+1} \bar{K}$ with these constraints, only two terms remain, namely,

$$\begin{aligned} \nabla_k \bar{K} - \nabla_{k+1} \bar{K} &= -\kappa(r_{k+1,k} + r_{k+1,k}) \\ &= -2\kappa \bar{s}'(0) S_k(t_k) > 0. \end{aligned} \quad (26)$$

Because $\bar{s}'(0) < 0$, we therefore have $\nabla_k \bar{K}_N > \nabla_{k+1} \bar{K}_N$. This inequality is valid at any point \mathbf{t}_N , provided that $t_{k+1} = t_k$, and in particular at \mathbf{t}_N^* , where we have $\nabla_{k+1} \bar{K}_N > 0$. We thus obtain that $\nabla_k \bar{K} > \nabla_{k+1} \bar{K} > 0$ at the minimum of \bar{K} , which is in contradiction with the original assumption.

We therefore conclude that if $t_k^* = t_{k-1}^*$ then for all $j < k$ we must have $t_k^* = t_j^* = 0$: in the optimal strategy, if searchers are launched simultaneously it must necessarily be at the start of the process and not later.

From this conclusion, we can now identify the optimal number of agents, N_s , that should be launched simultaneously at $t = 0$. To do this, we note that the optimal strategy must be such that $\nabla_p \bar{K}_N \geq 0$ for $p \leq N_s$ and $\nabla_p \bar{K}_N = 0$ for $p > N_s$. We will therefore calculate the derivative $\nabla_k \bar{K}$ for an arbitrary k with two constraints, namely, $t_p = t_k = 0$ for $p \leq k$ and $\nabla_p \bar{K}_N = 0$ for $p > k$. If this derivative is positive, we know from Eq. (26) that all previous derivatives for $p < k$ are also positive. We therefore identify N_s as the largest value of k for which $\nabla_k \bar{K}$ is positive under the mentioned constraints.

We first evaluate $\nabla_k \bar{K}$ with $t_2 = \dots = t_k = 0$ using Eq. (25) and note that some terms can be simplified, namely,

$$s_k = 1, \quad (27)$$

$$I_{nk} = \int_{t_n}^{t_{n+1}} \bar{s}'(t) \bar{s}(t)^{k-1} \prod_{p=k+1}^n s(t - t_p) dt \quad \text{for } n > k, \quad (28)$$

$$\sum_{n=1}^{k-1} r_{kn} = (k-1) \bar{s}'(0) \quad \text{for } n > k. \quad (29)$$

The integral term I_{nk} can be transformed using integration by parts, yielding

$$I_{nk} = \frac{1}{k} [s_{n+1} - s_n] - \frac{1}{k} \sum_{p=k+1}^n I_{np}. \quad (30)$$

Now, summing over all values of $n \geq k$ yields

$$\begin{aligned} \sum_{n=k}^N (1 + n\gamma) I_{nk} &= -\frac{1}{k} - \gamma - \frac{\gamma}{k} \sum_{n=k+1}^N s_n \\ &\quad - \frac{1}{k} \sum_{p=k+1}^N \sum_{n=p}^N (1 + n\gamma) I_{np}, \end{aligned} \quad (31)$$

where we have used $s_k = 1$ and $s_{N+1} = 0$. We now use the condition $\nabla_p \bar{K} = 0$ for $p > k$, which can be written using Eq. (25) as

$$\gamma s_p + \sum_{n=p}^N (1 + n\gamma) I_{np} - \kappa \left(\sum_{n=1}^{p-1} r_{pn} - \sum_{n=p+1}^N r_{np} \right) = 0. \quad (32)$$

Now, we note that the summand in the last term of Eq. (31) is exactly the second term in the gradient of $\nabla_p \bar{K}$ in Eq. (32).

Summing Eq. (32) over $p > k$ yields

$$\begin{aligned} \sum_{p=k+1}^N \sum_{n=p}^N (1 + n\gamma) I_{np} &= - \sum_{p=k+1}^N \gamma s_p + \kappa \sum_{p=k+1}^N \\ &\quad \times \left(\sum_{n=1}^{p-1} r_{pn} - \sum_{n=p+1}^N r_{np} \right). \end{aligned} \quad (33)$$

By swapping the order of summation in the terms proportional to κ we can show

$$\sum_{p=k+1}^N \left(\sum_{n=1}^{p-1} r_{pn} - \sum_{n=p+1}^N r_{np} \right) = k \sum_{p=k+1}^N r_{pk}, \quad (34)$$

where we have used the fact that $r_{pn} = r_{pk}$ for $n \leq k$. We therefore obtain

$$\sum_{p=k+1}^N \sum_{n=p}^N (1 + n\gamma) I_{np} = - \sum_{p=k+1}^N \gamma s_p + k\kappa \sum_{p=k+1}^N r_{pk}. \quad (35)$$

Now, substituting Eq. (35) into Eq. (31) and inserting Eqs. (27), (29), and (31) into Eq. (25) finally yields

$$\nabla_k \bar{K} = \frac{1}{k} + \kappa(k-1) \bar{s}'(0). \quad (36)$$

N_s is therefore found as the largest value of k for which this quantity is positive, i.e.,

$$N_s = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{2} - \frac{1}{\kappa \bar{s}'(0)}} \right\rfloor \underset{|\kappa \bar{s}'(0)| \rightarrow 0}{\approx} \frac{1}{\sqrt{-\kappa \bar{s}'(0)}}. \quad (37)$$

Surprisingly, N_s does not depend on γ at all. No matter how much it costs to sustain a searcher, the number of agents to be introduced into the system at $t = 0$ will only be governed by the launch cost κ and the initial slope of s' , i.e., the initial value of the first-passage time distribution: if $\bar{s}(t)$ is very sharply decreasing at short times, i.e., if the probability to find the target quickly is high, there is no interest in launching multiple searchers initially as the benefit in the search time would be overcompensated by the launch cost.

C. Numerical optimization

1. Test cases

While the statistical properties of the search cost depend on the details of the SASP, in a vast majority of single-agent search processes $\bar{s}(t)$ has a functional form that falls into only a few different classes. At long times $\bar{s}(t)$ either decays exponentially (e.g., in confined domains [42,43]) or algebraically (e.g., in open space [44]). Faster decays are extremely rare. Note that we consider here only SASPs whose probability to eventually find the target is 1, i.e., $\lim_{t \rightarrow \infty} \bar{s}(t) = 0$. At short times, three cases can be distinguished. First, the initial slope of $s'(t)$ can be very low and asymptotically close to zero. This happens for searches that cannot be infinitely fast, e.g., when the target and the searchers are initially always at a finite distance from one another, and leads to nonmonotonic—and

nonconvex—first-passage-time distributions [45,46]. Second, $-s'(0)$ is finite but sufficiently large for $\bar{s}(t)$ to be convex. This is a frequent feature of search processes with random initial positions of the searchers [47]. Third, $s'(t)$ decays infinitely fast at $t = 0$ and the first-passage-time distribution diverges at $t = 0$, as it is observed in some simple diffusive processes [48].

In order to reach a broad understanding of possible optimal launch strategies, we construct test cases which combine specific short- and long-time behaviors. To do this, we compose $t \mapsto e^{-\lambda t}$ and $t \mapsto (1 + \lambda\theta^{-1}t)^{-\theta}$ characterizing the long-time behavior with either $x \mapsto \sin(\pi x/2)$, $x \mapsto 1$ or $x \mapsto 2 \arcsin(x)/\pi$ that characterize the short-time behaviors. The six resulting SASPs we focus on are therefore

$$s_{\text{exp}}^{\text{mild}}(t) = e^{-\lambda t}, \quad (38)$$

$$s_{\text{exp}}^{\text{flat}}(t) = \sin\left(\frac{\pi}{2}e^{-\lambda t}\right), \quad (39)$$

$$s_{\text{exp}}^{\text{sharp}}(t) = \frac{2}{\pi} \arcsin(e^{-\lambda t}), \quad (40)$$

$$s_{\text{alg}}^{\text{mild}}(t) = \frac{1}{(1 + \lambda t \theta^{-1})^\theta}, \quad (41)$$

$$s_{\text{alg}}^{\text{flat}}(t) = \sin\left(\frac{\pi}{2(1 + \lambda t \theta^{-1})^\theta}\right), \quad (42)$$

$$s_{\text{alg}}^{\text{sharp}}(t) = \frac{2}{\pi} \arcsin\left(\frac{1}{(1 + \lambda t \theta^{-1})^\theta}\right). \quad (43)$$

In practice we used $\theta = 2$. We show these six functions in Fig. 2 for visualization, as well as the mean search cost \bar{K} as a function of t_2 in a two-searcher process.

To identify optimal launch strategies, we performed standard gradient descent optimization. We start with $N = 2$ and we initialize t_2 to a random value $t_2 = t_2^{(0)}$. At each step, we calculate the gradient of \bar{K} with respect to t_2 and update $t_2^{(n+1)} = t_2^{(n)} - r \partial_{t_2} \bar{K}|_{t_2=t_2^{(n)}}$, where r is the learning rate. We stop the iteration either if t_2 has converged to a constant value or if $t_2 = 0$ and $\partial_{t_2} \bar{K} > 0$. We then add a new degree of freedom, t_3 , that we initialize to a random value $t_3^{(0)} > t_2$. At each step we now compute the full gradient of \bar{K} with respect to t_2 and t_3 , which we use to update both of the degrees of freedom until convergence. We continue this process by adding more and more degrees of freedom until we reach $N - 1$ degrees of freedom that are such that $\bar{S}(t_N) < 0.001$.

Among the six test cases under study, four of them had unique local minima of the mean search cost, namely, the *mild* and *sharp* cases. They correspond to log-convex SASPs, which is consistent with the prediction for two searchers. No local maximum was found such that Eq. (37) holds for these cases. The two other cases, i.e., nonconvex SASPs, led to more complex structures for the mean search cost and are analyzed separately.

2. Convex SASP

We first start with the convex test cases. Before we proceed with the numerical optimization, we first treat the mild

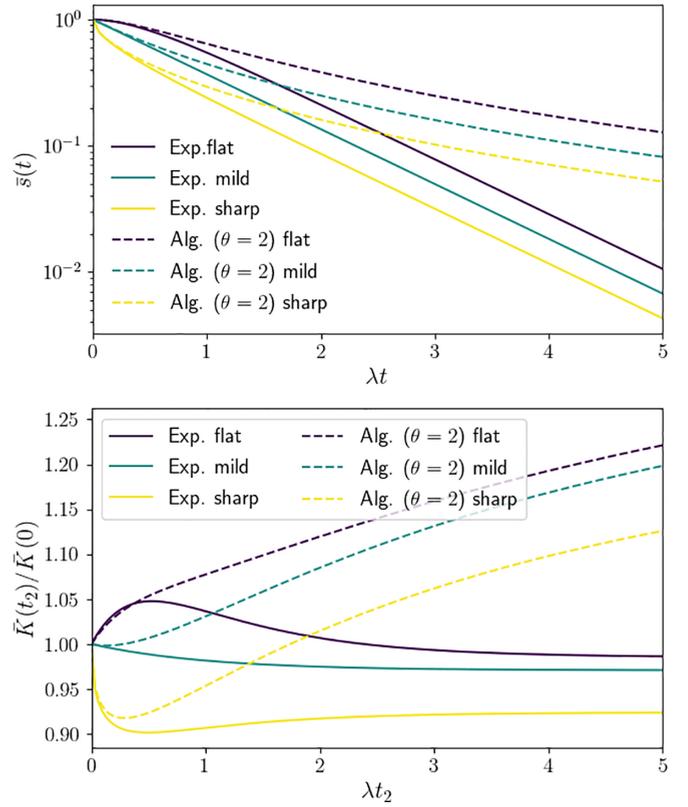


FIG. 2. Top: Test cases of single-agent survival probabilities for the numerical optimization of the mean search cost. They combine two long-time with three short-time behaviors and allow to discriminate between possible optimal launch strategies. Bottom: Mean search cost \bar{K} as a function of t_2 , normalized by its value for $t_2 = 0$, for the six test cases of SASP. We use here $\kappa\lambda = 0.55$ and $\gamma = 0.1$. Some SASPs lead to nontrivial minima for the mean cost while others are minimized for $t_2 = 0$ or $t_2 \rightarrow \infty$.

exponential case $\bar{s}(t) = e^{-\lambda t}$ separately as it can be fully analyzed analytically. In this case we have

$$\bar{T} = \lambda^{-1} \left[1 - \sum_{n=2}^{\infty} \frac{e^{-\lambda \sum_{k=1}^{n-1} t_n - t_k}}{n(n-1)} \right], \quad (44)$$

$$\bar{T} = \lambda^{-1}, \quad (45)$$

$$\bar{N} = 1 + \sum_{n=2}^{\infty} e^{-\lambda \sum_{k=1}^n (t_n - t_k)}. \quad (46)$$

Now, note that $\sum_{k=1}^n (t_n - t_k) = \sum_{l=1}^{n-1} l \Delta_l$, where $\Delta_k = t_{k+1} - t_k$ is the time interval between two consecutive launches. \bar{K} can thus be fully expressed as a function of these time intervals and its partial derivative with respect to Δ_k reads then

$$\frac{\partial \bar{K}}{\partial \Delta_k} = \lambda \kappa \sum_{n=k+1}^{\infty} e^{-\lambda \sum_{l=1}^{n-1} l \Delta_l} \left(\frac{\lambda^{-1}}{n(n-1)} - \kappa \right). \quad (47)$$

Here, the terms for which $\kappa\lambda n(n-1) < 1$ contribute positively to the gradient. Let $N_s = \lfloor \frac{1}{2}(1 + \sqrt{1 + \frac{4}{\kappa\lambda}}) \rfloor$ be the lowest value of n such that the latter equality is verified. Then, for any $k \geq N_s$ the derivative $\partial_{\Delta_k} \bar{K}$ is negative such that

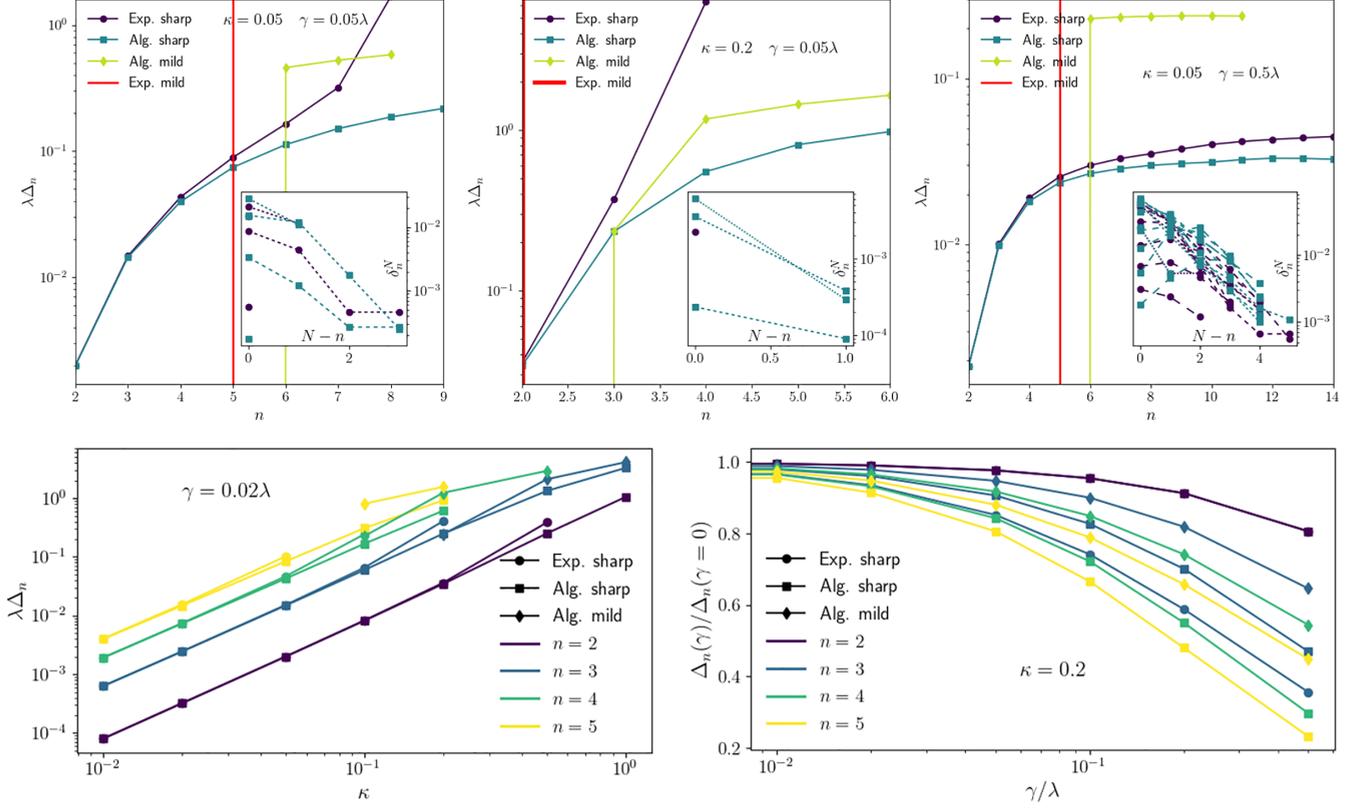


FIG. 3. Top: Launch intervals Δ_n normalized by λ in the optimal strategies, as identified by numerical optimization, as a function of the index of the n th agent. Different panels correspond to different values of κ and γ while different curves correspond to different SASPs. The inset shows the relative variation δ_n^N as a function of $N-n$, for which $\Delta_n^{N+1} - \Delta_n^N$ was found larger than the tolerance of the gradient descent algorithm. We find that the optimal launch interval for low n is governed by the short-time behavior of $\bar{s}(t)$. For mild ones, multiple searchers may be launched initially, which is shown here with vertical lines as $\Delta_n = 0$ for $n \leq N_s$. For sharp ones, only one searcher is launched at $t = 0$ and subsequent searchers are launched at later times. For large n , the launch interval is governed by the long-time behavior of $\bar{s}(t)$. For algebraic ones, Δ_n reaches a constant value, while for exponential ones, Δ_n diverges. We do not observe this divergence on the last plot as it occurs at very large values of n . Bottom: Optimal introduction intervals Δ_n as a function of κ (left) and γ (right) for three test cases. The values are normalized by Δ_n for $\gamma = 0$ in the second plot for better visualization. We find that optimal launch intervals increase with κ but decrease with γ .

the optimal strategy is such that $\Delta_k \rightarrow \infty$. In this limit, the contributions of all terms with $n \geq N_s$ vanish because of the exponential factor. Therefore, for all $k < N_s$ the gradient will be positive such that the cost will be minimized for $\Delta_k = 0$. The optimal strategy is therefore such that N_s searchers should be launched at $t = 0$ and none later. In this case, the mean search cost is found as

$$\bar{K}_{\text{opt}} = \gamma + \sqrt{\kappa\lambda} \left(\frac{2}{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}} + \frac{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}}{2} \right), \quad (48)$$

which is obtained by neglecting the floor part in N_s . As $\kappa \rightarrow 0$, the terms between brackets tend to 1 and the overall cost grows as $\sqrt{\kappa\lambda}$. However, as $\kappa\lambda \gg 1$, the brackets are dominated by the second term, which tends to $\sqrt{\kappa\lambda}$, making the cost grow linearly with $\kappa\lambda$.

For the three other convex test cases, we perform the analysis numerically. First we investigate to which extent the total number of available agents impacts the optimal introduction times by defining the relative variation $\delta_n^N = 1 - \Delta_n^{N+1}/\Delta_n^N$ upon having a new available searcher, where the superscript

N refers to the total number of searchers. As shown in the insets of Fig. 3, δ_n^N decays roughly exponentially with $N-n$ starting with a relatively low amplitude. The optimal launch times t_n obtained in the limit $N \rightarrow \infty$ are therefore a good approximation of the actual optimal ones for a finite number N of available searchers.

In the limit $N \rightarrow \infty$, our results allow to identify the optimal launch strategies and to classify them, as shown in the top panel of Fig. 3. We observe that the short-time behavior governs the launch time of the first agents while the long-time behavior governs the introduction of later agents. For the initially *mild* SASP, we have $N_s \geq 1$ in accordance with Eq. (37); i.e., multiple walkers may optimally be introduced simultaneously at the start of the process. For the *sharp* SASP, it holds on the contrary that $N_s = 1$: one necessarily has to wait a certain time before launching a second walker. For searchers introduced at later times, the optimal launch intervals Δ_n diverge for exponentially decaying SASP: as one adds more and more searchers in the system, the probability to find the target in a short amount of time becomes so high that it would cost more to introduce a new searcher quickly than to simply wait for the ones already launched to find the target.

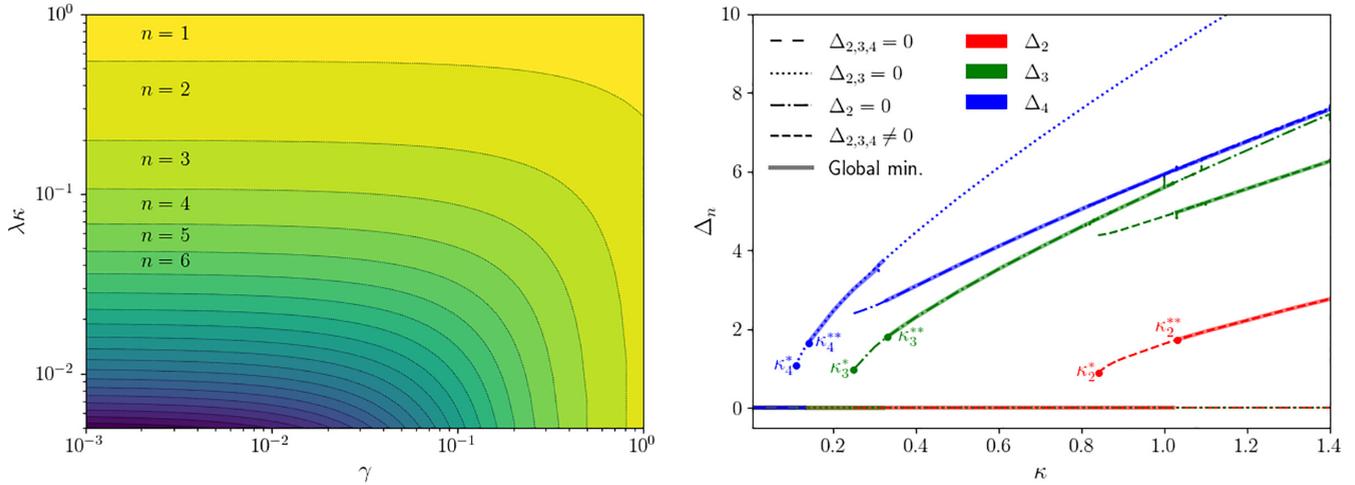


FIG. 4. Optimal launch strategies for nonconvex SASP. Left: In the case of an exponential decay, the best strategy is to launch n searchers initially and none after. We show the corresponding optimal number of searchers as a function of κ and γ . Right: For the algebraic decay, finite optimal launch times are observed. We show all launch intervals Δ_n that are locally optimal as a function of κ for $N = 4$ and $\gamma = 0.01\lambda$. Different dashed line styles indicate different local minima while the solid line indicates the global one. Colors indicate the second, third, and fourth searchers.

On the contrary, Δ_n reaches a constant value as n grows for algebraically decaying SASP. Here, even as one adds new searchers in the system, the probability to wait a long time may still remain non-negligible. It is therefore preferable to launch searchers at a constant pace even if it is at a certain cost.

The number of agents around which the transition between *early* and *late* agents occurs decreases with κ and increases with γ and roughly corresponds to introduction times t_n such that $\lambda t_n \sim 1$. This is consistent with the dependence of the optimal launch intervals with the cost contributions; namely, it increases with κ and decreases with γ as shown in the lower panel of Fig. 3. While the dependence on κ is not surprising as a large launch cost should reward a longer wait before launching a new searcher, the dependence on γ is less intuitive. It is in fact preferable to launch searchers at a higher frequency when the *sustaining* rate γ is larger, indicating that the gain in the overall search time overcompensates the larger rate of resources required to sustain the new searchers. This trend holds for all tested SASPs and appears to be a general result for a wide variety of search processes.

3. Nonconvex case

The case of nonconvex SASP is more complex to treat as multiple local minima may exist. A first straightforward *locally* optimal strategy can be identified using Eq. (36). This equation still holds for nonconvex SASP but the conclusion that we draw from it is different. In fact, if $s'(0) > -\frac{1}{\kappa k(k-1)}$, the gradient $\nabla_k \bar{K}$ for $t_2 = \dots = t_k = 0$ is negative, provided that the derivatives with respect to later launch times vanish. This is in particular true for any k if $s'(0) = 0$. There, for any $k \leq N$, there exists a *locally* optimal strategy that consists in launching k searchers initially and the next searchers later; i.e., \bar{K} is at a local minimum at this location. To rationalize this, we note that since the probability for the first searcher to find the target at very short times is low, there is no gain in

waiting a short amount of time for launching next searchers compared to launching it together with the first one. However, this locally optimal strategy is not necessarily the globally best one, especially if the launch cost κ is high.

For the exponential test case $s_{\text{exp}}^{\text{flat}}$, we observe that all local minima are such that $t_i = 0$ for $i \leq k$ and $t_i \rightarrow \infty$ for $i > k$: similarly to the mild exponential case, the optimal strategy is to launch a certain number of searchers initially and none after. However, this optimal number of initial searchers is not given by Eq. (37) but can be found numerically by minimizing the mean search cost of the form $n\kappa + (1 + n\gamma) \int_0^\infty \bar{s}(t)^n dt$ with respect to n . We show the resulting optimal number of searchers to be launched in the (κ, γ) plane in Fig. 4 and note that it does depend on γ , unlike the convex case.

In the case of an algebraic decay, i.e., the test case $s_{\text{alg}}^{\text{flat}}$, we observe that the situation is more complex as finite, nonzero launch times can be optimal. We show in Fig. 4 an example of the optimal strategies for $N = 4$. For low values of κ , the only local minimum is located at $\Delta_{2,3,4} = 0$: the optimal strategy is to launch all searchers simultaneously. As one progressively increases κ , a local minimum with $\Delta_4 > 0$ appears at $\kappa = \kappa_4^*$ but only becomes the global minimum for $\kappa = \kappa_4^{**} > \kappa_4^*$. Increasing again κ , local minima with $\Delta_{2,3} > 0$ appear at $\kappa_{2,3}^*$ and again become the global minima at $\kappa_{2,3}^{**}$. Because of the high number of local minima, the numerical optimization for a larger number of searchers becomes challenging. However, one can legitimately expect from our results in the convex case that the launch time of later searchers should be governed by the long-time behavior of the SASP, which is here algebraic. We thus conjecture that the time interval between consecutive launches should optimally reach a constant value, similarly to the convex SASPs that decay algebraically.

IV. OPTIMIZING THE QUANTILES

Beyond the mean search cost, one plausible criterion for optimizing the search strategy may be to minimize the

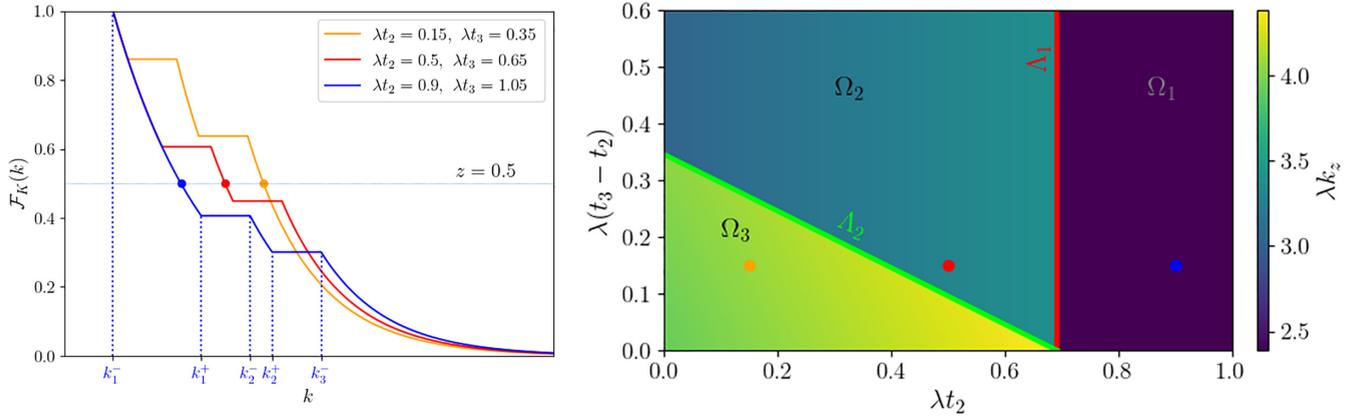


FIG. 5. Quantile optimization for $N = 3$ and $\bar{s}(t) = e^{-\lambda t}$ with $\gamma = 1$ and $\kappa\lambda = 1$. Left: We show \mathcal{F}_K as a function of k for various choices of t_2 and t_3 . The dashed line indicates $z = 0.5$, i.e., the median. Each of the curves is located in a different subdomain Ω_n in the (t_2, t_3) plane. Right: Median cost in the (t_2, t_3) plane for the same values of κ and γ . The boundaries Λ_1 and Λ_2 are shown as the solid lines and the three dots correspond to the three curves shown in the left panel. Here, the minimum of k_z is located on Λ_1 defined as $\lambda t_2 = -\ln(z)$, where k_z is constant. The optimal strategy consists here in launching the first searcher and waiting a time at least longer than $-\ln(z)$ to launch the second and third searchers in order to minimize the quantile.

probability that the search cost reaches too large values. This can be quantified through the q quantile of the search cost distribution. Given the probability distribution $r_K(k)$ of the search cost, the q quantile \hat{k}_q is defined as the solution of

$$\int_0^{\hat{k}_q} r_K(k) dk = \frac{1}{q}. \quad (49)$$

In particular \hat{k}_2 is known as the median cost. Denoting $\mathcal{F}_K(k)$ the probability that the search cost is greater than k , Eq. (49) can also be written as $\mathcal{F}_K(\hat{k}_q) = 1 - q^{-1}$. Let us now note $z = 1 - q^{-1}$ and let k_z be the solution of the equation

$$\mathcal{F}_K(k_z) = z. \quad (50)$$

With these notations, the probability that the search cost is larger than k_z is z . Technically, we have $k_z = \hat{k}_{\frac{1}{1-z}}$, i.e., k_z is the $(\frac{1}{1-z})$ quantile. Let us now show how to minimize it for a given value of z .

A. Minimizing a single q quantile

Because the search cost K is a piecewise affine function of the first-passage time T , we can straightforwardly relate \mathcal{F}_K to \bar{S} and in turn to \bar{s} via

$$\mathcal{F}_K(k) = \begin{cases} \prod_{i=1}^n \bar{s}(\chi_n(k) - t_i) & \text{if } k_n^- \leq k < k_n^+ \\ \prod_{i=1}^n \bar{s}(\chi_n(k_n^+) - t_i) & \text{if } k_n^+ \leq k < k_{n+1}^-, \end{cases} \quad (51)$$

where we have defined

$$\chi_n(k) = \frac{k - n\kappa + \gamma \sum_{k=2}^n t_k}{1 + n\gamma}, \quad (52)$$

$$k_n^+ = n\kappa + (1 + n\gamma)t_{n+1} - \gamma \sum_{k=2}^n t_k, \quad (53)$$

$$k_n^- = k_{n-1}^+ + \kappa. \quad (54)$$

$\mathcal{F}_K(k)$ consists in an alternation of monotonically decreasing parts and of constant plateaus corresponding to the launch of a new searcher, as shown in Fig. 5.

Let Ω be the $(N - 1)$ -dimensional space of possible values taken by $\{t_2, \dots, t_N\}$. Because $\mathcal{F}_K(k)$ is a piecewise function of k whose intervals are functions of the times t_i , the solution k_z of $\mathcal{F}_K(k_z) = z$ is also a piecewise function in Ω . We note Ω_n the subspace of Ω such that $k_z \in [k_n^-, k_n^+]$. Subdomains Ω_n and Ω_{n+1} are adjacent; i.e., they share a border Λ_n defined as $k_z = k_n^+$, or equivalently $k_z = k_{n+1}^-$. A visual representation of these subdomains is shown in Fig. 5 (right-hand panel).

In Ω_n , k_z is found as the solution of

$$\prod_{i=1}^n \bar{s}(\chi_n(k_z) - t_i) = z. \quad (55)$$

If there is a local minimum of k_z in Ω_n it must be such that $\partial_i k_z = 0$ for all $i \leq n$. Differentiating this equation with respect to t_i we find after some algebra

$$(\partial_i k_z + \gamma) \sum_{j=1}^n \xi_j^{(n)} - (1 + n\gamma) \xi_i^{(n)} = 0, \quad (56)$$

where we have defined

$$\xi_i^{(n)} = \frac{\bar{s}'(\chi_n(k_z) - t_i)}{\bar{s}(\chi_n(k_z) - t_i)}. \quad (57)$$

Noting $\langle \xi^{(n)} \rangle = \frac{1}{n} \sum_{j=1}^n \xi_j^{(n)}$ the mean of all $\xi_j^{(n)}$, the condition $\partial_i k_z = 0$ for all $i \leq n$ for the existence of a local minimum in Ω_n imposes

$$\xi_i^{(n)} = \langle \xi^{(n)} \rangle \frac{n\gamma}{1 + n\gamma} \geq \langle \xi^{(n)} \rangle. \quad (58)$$

This cannot be realized for all i as all $\xi_j^{(n)}$ cannot simultaneously be larger than their mean, which implies that there cannot be a local minimum within Ω_n . k_z is therefore minimized at the boundaries of Ω_n , i.e., one of the lower-dimensional subspaces Λ_n . There, the function $k_z(\{t_i\})$ is discontinuous, jumping from a value k_n^+ to a larger value k_{n+1}^- . The global minimum will therefore be such that $k_z = k_n^+$ for a certain value of n .

Using the definition of $k_n^+ = n\kappa + (1 + n\gamma)t_{n+1} - \gamma \sum_{k=1}^n t_k$ and substituting it into Eq. (51), the equation that

defines the $(N - 2)$ -dimensional subspace Λ_n reads

$$\prod_{i=1}^n \bar{s}(t_{n+1} - t_i) = z. \quad (59)$$

This constrains the value of one of the t_i and we can choose in particular to express $t_n + 1$ as

$$t_{n+1} = \bar{s}^{-1} \left(\frac{z}{\prod_{i=2}^n \bar{s}(t_{n+1} - t_i)} \right), \quad (60)$$

where we have used $t_1 = 0$. Substituting this expression into the definition of k_n^+ yields

$$k_n^+ = n\kappa + (1 + n\gamma)\bar{s}^{-1} \left(\frac{z}{\prod_{i=2}^n \bar{s}(t_{n+1} - t_i)} \right) - \gamma \sum_{i=2}^n t_i. \quad (61)$$

If there exists a local minimum in Λ_n , then it should be such that $\partial_i k_n^+ = 0$ for all $i \leq n$. After some algebra, we can write $\partial_i k_n^+$ as

$$\partial_i k_n^+ = (1 + \gamma) \frac{\bar{s}'(t_{n+1} - t_i) \bar{s}(t_{n+1})}{\bar{s}(t_{n+1} - t_i) \bar{s}'(t_{n+1})} - \gamma. \quad (62)$$

If \bar{s} is logarithmically convex, then \bar{s}'/\bar{s} is a negative, monotonically growing function, leading to $\partial_i k_n^+ > 1$. We therefore conclude that k_n^+ decreases monotonically with each t_i such that there cannot be any local extremum of k_z within Λ_n . The minimum of k_z has thus to be found for $t_i = 0$.

To find the global minimum of k_z we finally need to compare k_n^+ for all values of n where we set $t_i = 0$ for $i \leq n$ and $t_{n+1} = \bar{s}^{-1}(z^{\frac{1}{n}})$ as imposed by Eq. (59). This yields

$$k_n^+ = n\kappa + (1 + n\gamma)\bar{s}^{-1}(z^{\frac{1}{n}}). \quad (63)$$

We finally conclude that the optimal strategy that minimizes the q quantile with $q = 1 - z^{-1}$ is therefore to initially launch the $\tilde{N}_s(z)$ searcher where n minimizes the right-hand side of Eq. (63) and launch the next searchers at times later than

$$\tilde{t}_s(z) = \bar{s}^{-1}(z^{\frac{1}{\tilde{N}_s(z)}}). \quad (64)$$

This general result holds for any log-convex SASP.

B. Subsequent quantiles

Following the optimal strategy that minimizes k_z as described in the previous paragraph, we note that the launch times t_i for $i > \tilde{N}_s(z)$ do not impact the optimized quantile as long as they are larger than t_z^* . One is therefore free to launch subsequent searchers arbitrarily without altering the result presented in the previous section. One might in particular want to optimize their launch times in order to minimize a subsequent q' quantile with $z' = 1 - q'^{-1} < 1 - q^{-1} = z$, provided that the q quantile has already been optimized.

We can perform the same analysis as in the previous section by setting all $t_i = 0$ for $i \leq \tilde{N}_s(z)$ and check whether a local minimum with respect to later launch times can be found within one of the subdomains Ω_n with $n \geq n_z^*$. Following similar calculations, we show that the condition $\partial_i k_{z'} = 0$ within a subdomain Ω_n with $i \geq n \geq \tilde{N}_s(z)$ leads to

$$\xi_i^{(n)} = \frac{\gamma}{1 + n\gamma} \left[(n - \tilde{N}_s(z)) \langle \xi^{(n)} \rangle_{\tilde{N}_s(z)}^n + \tilde{N}_s(z) \xi_z^{(\tilde{N}_s(z))} \right], \quad (65)$$

where $\langle \xi^{(n)} \rangle_{\tilde{N}_s(z)}^n$ is the mean of the $\xi_j^{(n)}$ for $\tilde{N}_s(z) + 1 \leq j \leq n$. Using again the logarithmic convexity of \bar{s} , we have $\langle \xi^{(n)} \rangle_{\tilde{N}_s(z)}^n \geq \xi_z^{(\tilde{N}_s(z))}$. Because $\xi_i^{(n)} \leq 0$ for any i , multiplying the equation with -1 and using the latter inequality yields

$$|\xi_i^{(n)}| \geq \frac{n\gamma}{1 + n\gamma} \langle |\xi^{(n)}| \rangle_{\tilde{N}_s(z)}^n \geq \langle |\xi^{(n)}| \rangle_{\tilde{N}_s(z)}^n. \quad (66)$$

Similarly to Eq. (58), this cannot be realized for all $i > \tilde{N}_s(z)$ and we conclude that there cannot be a local minimum of $k_{z'}$ within one of the subdomains Ω_n . The minimum must therefore be found in one of the boundaries Λ_n . In this case, following the same line of calculations that led to Eq. (62), we obtain

$$\partial_i k_n^+ = \gamma - \frac{1 + \gamma \bar{s}'(t_{n+1} - t_i) \bar{s}(t_{n+1})}{\tilde{N}_s(z) \bar{s}(t_{n+1} - t_i) \bar{s}'(t_{n+1})}. \quad (67)$$

Here, we cannot conclude in general on the sign of the latter quantity. Using the logarithmic convexity of \bar{s} we can simply bound $\partial_i k_n^+$ from above by $\gamma(1 - \tilde{N}_s(z)^{-1}) - 1$, which can take positive values if γ is large enough.

However, if $\gamma < (1 - \tilde{N}_s(z)^{-1})^{-1}$, $k_{z'}$ cannot be minimized within any of the Λ_n and its minimum is found for $t_i = t_z^*$ for all $\tilde{N}_s(z) < i \leq n$. In this case, the optimal strategy is therefore as follows. After having launched $\tilde{N}_s(z)$ searchers at $t = 0$ to minimize k_z , $\tilde{N}_s(z, z')$ new searchers should be launched at time $\tilde{t}_s(z)$ to minimize $k_{z'}$, where $\tilde{N}_s(z, z')$ is defined as

$$\tilde{N}_s(z, z') = \operatorname{argmin}_n [n\kappa + (1 + n\gamma) \tilde{t}^{(n)}(z, z')] - \tilde{N}_s(z) \quad (68)$$

and $\tilde{t}^{(n)}(z, z')$ is defined through Eq. (59) that in this situation reads

$$\bar{s}^{\tilde{t}^{(n)}(z, z') - \tilde{t}_s(z)} (z, z')^{n - \tilde{N}_s(z)} \bar{s}^{\tilde{t}^{(n)}(z, z')}^{\tilde{N}_s(z)} = z'. \quad (69)$$

Later searchers should then be introduced at a time later than $\tilde{t}_s(z, z') = \tilde{t}^{\tilde{N}_s(z, z') + \tilde{N}_s(z)}(z, z')$. The procedure can then be repeated to again minimize subsequent quantiles.

C. Canonical example: The exponential SASP

Let us illustrate the results of the previous sections in a specific example, namely, for an exponentially decaying SASP $\bar{s}(t) = e^{-\lambda t}$, although similar calculations can still be performed for any log-convex SASP.

Here, the boundaries Λ_n defined through Eq. (59) are found as

$$t_{n+1} = \frac{1}{n} \left[\sum_{k=1}^n t_k - \lambda^{-1} \ln(z) \right]. \quad (70)$$

We show them as lines in the (t_2, t_3) plane for $N = 3$ in Fig. 5. There, the minimum of k_z is found either in the Ω_1 subdomain where k_z is constant, or on the line $t_2 = 0$ in Ω_2 as k_z does not depend on t_3 there, or at $t_2 = t_3 = 0$. In either of these locations, we have $k_z = k_n^+$ with

$$k_n^+ = n\kappa - \left(\frac{1}{n} + \gamma \right) \ln(z). \quad (71)$$

The value $\tilde{N}_s(z)$ that minimizes this quantity and the corresponding launch time t_z^* are found as

$$\tilde{N}_s(z) = \sqrt{\frac{-\ln(z)}{\lambda\kappa}}, \quad \tilde{t}_s(z) = \sqrt{\frac{-\kappa \ln(z)}{\lambda}}. \quad (72)$$

Optimally, $\tilde{N}_s(z)$ searchers should be introduced at $t = 0$ while the next searchers should be introduced at a time later than t_z^* . Interestingly, this result does not depend at all on γ . Whatever the cost for sustaining the searchers, it will not impact how many should be launched initially in the optimal strategy.

When it comes to launching later searchers in order to minimize a subsequent quantile k_z , we first need to evaluate Eq. (67), reading $\partial_i k_n^+ = (1 + \gamma)/\tilde{N}_s(z) - \gamma$ for $i \leq n < \tilde{N}_s(z)$. If $\gamma > \tilde{N}_s(z)/(\tilde{N}_s(z) - 1)$, then $\partial_i k_n^+ < 0$ and the minimum is found for $t_i = t_{n+1}$. In this case, k_n^+ increases linearly with n such that the value $n \geq \tilde{N}_s(z)$ that minimizes it is $n = \tilde{N}_s(z)$: optimally no new searcher should be introduced. On the other hand, if $\gamma < \tilde{N}_s(z)/(\tilde{N}_s(z) - 1)$, the minimum of k_z in each Λ_n is found for $t_i = \tilde{t}_s(z)$. In this case, Eq. (69) reads $\tilde{t}^{(n)}(z, z') = \tilde{t}_s(z)(1 - \tilde{N}_s(z)/n) - \ln(z')/n\lambda$ and we can show that k_n^+ is minimized for a value

$$m_s(z, z') = \sqrt{\frac{-\ln(z) - \ln(z')}{\kappa\lambda + \gamma\sqrt{-\kappa\lambda \ln(z)}}}. \quad (73)$$

If this number is larger than $\tilde{N}_s(z)$ one should launch $\tilde{n}_s(z, z') = m_s(z, z') - \tilde{N}_s(z)$ new searchers at time $t \approx (z, z')$. Otherwise, we launch no new searcher at time $t \approx (\tilde{N}_s(z) + 1)(z, z')$. Again, we can then continue to look for optimal ways to launch the next searchers and minimize subsequent quantiles.

V. PROOF OF PRINCIPLE

In order to verify the validity of our results, we now compare our prediction with data from numerical simulations in an example of a search process. We consider a Lévy flight process, a class of random walks that has been extensively studied for target searches and which are generally known to lead to lower first-passage times than random walks with short-ranged jumps [49]. In a two-dimensional square box of size L with periodic conditions, we place a target at the origin \mathbf{r}_0 . Searchers are initialized at random positions $\mathbf{r}_i = (x_i, y_i)$ where both x_i and y_i are drawn from a Gaussian distribution centered in \mathbf{r}_0 and with standard deviation $\sigma_{x,y}$. At each time step, the searcher jumps from its position \mathbf{r}_t to a new position $\mathbf{r}_{t+1} = \mathbf{r}_t + \Delta\mathbf{r}$ where $\Delta\mathbf{r} = l(\cos\theta, \sin\theta)$. θ is a random angle uniformly distributed between 0 and 2π while l is a random jump length drawn from a Pareto distribution $p_l(l) \propto l^{-\alpha}$ for $l > 1$ and $\alpha > 1$. The process stops whenever the searcher reaches a position \mathbf{r} such that $|\mathbf{r} - \mathbf{r}_0| < 1$; i.e., it has reached the vicinity of the target.

We start by running simulations for a single searcher using $\alpha = 5$, $L = 100$, and $\sigma_x = \sigma_y = 10$ and evaluate the SASP $\bar{s}(t)$, which we show in Fig. 6. At short times, it decays very sharply. As the probability for the searcher to be initialized

very close to the target is high, many trajectories are such that the searcher finds the target almost immediately after being launched, and thus the probability for the target to be not reached yet after a certain time decreases fast initially. At long times, we observe an exponential decay, which is also consistent with most SASPs of searches in closed domains. The corresponding single-search mean first-passage time is found to be $\bar{T}_1 = 19423$.

Based on this observation, our results from the previous section suggest that only one searcher should be launched initially, and later searchers should be launched at finite times with increasing consecutive launch intervals that eventually diverge. As an example, for $\kappa = 0.15\bar{T}_1$ and $\gamma = 0.01$, the numerical optimization performed in the previous section predicts that the optimal launch strategy for multiple searchers is obtained for $\Delta_2 = 0.015\bar{T}_1$, $\Delta_3 = 0.041\bar{T}_1$, and $\Delta_4 \rightarrow \infty$. In order to verify this prediction, we ran simulations with three searchers in which the second and third are launched at various times, and for each set of launch intervals (Δ_2, Δ_3) we computed the resulting MFPT. We plot it in Fig. 6 in the (Δ_2, Δ_3) plane in which we also indicate the location of the predicted optimal strategy and we observe a very good agreement as this predicted point coincides with the minimum obtained via numerical simulations.

Similarly, we compare the predicted optimal strategy to minimize the median cost (i.e., the quantile with $z = 0.5$). For the same values of κ and γ , our prediction is such that the second searcher should be launched at $t = 0$ and the third after $t = 0.22\bar{T}_1$, knowing that any later time does not impact the median cost. This prediction matches perfectly the simulation data, which also coincide very well with the predicted domain lines Λ_n defined and described in Sec. III, as shown in Fig. 6.

VI. COMPARISON WITH STOCHASTIC RESETTING

The process of launching a new searcher until a target has been found is reminiscent of search processes with stochastic resetting. Instead of launching new searchers at the initial position in intervals, one could instead reset the first searcher to its initial position at a certain rate. Such processes have attracted a lot of interest recently, both on the theoretical and experimental sides [24,50–53]. Fine tuning the time at which the searcher is subject to resetting can have a significant impact on the overall search efficiency [54–57]. Given the cost for resetting a searcher and sustaining it over time, we can legitimately wonder whether resetting the first searcher is more cost effective than launching a new one. While the time it takes to find the target will be reduced by launching a new searcher, the cost for sustaining multiple agents rather than a single one may overcompensate the gain in the overall search time.

Let us first define the search cost K_r of a resetting process in accordance with our search cost for the launch process, reading

$$K_r = (1 + \gamma_r)T + \kappa_r N_r, \quad (74)$$

where T is the first-passage time and N_r is the number of resetting events. Here $\gamma_r = \gamma$ is the sustaining rate of the searcher, and κ_r may vary from one trajectory to the next depending on the process. There exist, e.g., processes where

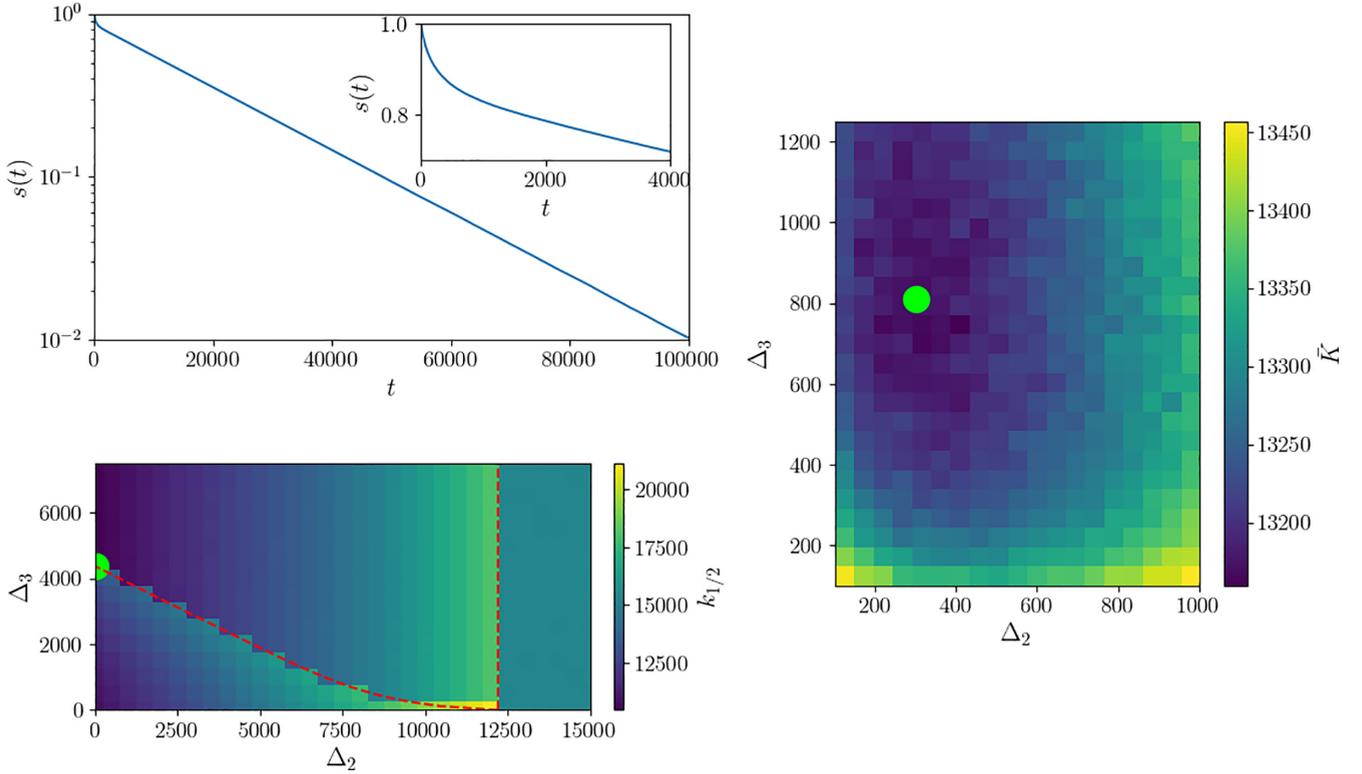


FIG. 6. Numerical simulations of a Lévy flight process with $\alpha = 5$ in which the target is placed at the origin of a box with periodic boundary conditions and searchers are launched from a position that is Gaussian distributed around this target. Top left: SASP $\bar{s}(t)$ as a function of t . The inset expands the short-time behavior. Bottom left: Median search cost for a three-searcher process as a function of Δ_2 and Δ_3 as sampled from the numerical simulations for $\kappa = 0.15\bar{T}_1$ and $\gamma = 0.01$. The dotted lines are the boundary lines Λ_1 and Λ_2 as predicted by Eq. (59) while the green dot is the location of the minimum median cost as predicted by our theory, which coincides with the minimum obtained by numerical simulations. Right: The same analysis is shown for the mean search cost. Again, the theoretical prediction for the minimum search cost matches the simulation data.

the cost for resetting a searcher depends on the distance that it needs to be displaced in order to place it back to its initial position. While it is unlikely to derive general relevant results for the comparison between launching and resetting strategies, we will treat a canonical example in the next lines, namely, the one-dimensional diffusive search, that the reader could then adapt to their favorite random search process.

We consider a one-dimensional infinite system. A target is placed a position $x_T > 0$, and searchers are initialized at $x = 0$ and evolve according to standard diffusive Brownian motion with diffusion coefficient D . In this case, it is well known that the SASP is found to be

$$\bar{s}(t) = \text{erf}(x_T/2\sqrt{Dt}). \quad (75)$$

When it comes to resetting the first searcher to its initial position, two resetting costs have been considered in the literature. Either the resetting is proportional to the work one needs to perform to reset which typically scales like the distance it has reached just before being reset, or it is constant. In Ref. [22], Chechkin *et al.* showed that in a resetting process where the time between two resetting events was not conditioned on anything and was drawn from a single probability distribution, the optimal strategy is to make this distribution as narrow as possible around a mean value Δ . Following the same methodology, we can show that the mean search cost \bar{K}_r is minimized

for a fixed resetting time interval Δ , in which case it reads

$$\bar{K}_r(\Delta) = \frac{(1 + \gamma_r)G(\Delta) + \bar{\kappa}_r(\Delta)}{F(\Delta)}, \quad (76)$$

where $F(\Delta) = 1 - s(\Delta)$ and $G(\Delta) = \int_0^\Delta s(\tau)d\tau$ and $\bar{\kappa}_r$ is the average of the resetting costs over all possible resetting events. In the case of a resetting cost proportional to the distance $|\Delta x|$ between the searcher and its initial position upon resetting as used, e.g., in Refs. [58,59], we obtain

$$\bar{\kappa}_r(\Delta) = f \frac{\int_{-\infty}^{x_T} dx |x| c(x, \Delta)}{\int_{-\infty}^{x_T} dx c(x, \Delta)}, \quad (77)$$

where f is the cost per unit length and $c(x, t)$ is the particle density in an experiment with absorbing boundary at $x = x_T$, i.e., $\int_{-\infty}^{x_T} c(x, t) dx = \bar{s}(t)$. In particular for the one-dimensional (1D) diffusion model, $c(x, t)$ is known to be [60]

$$c(x, t) = \frac{1}{\sqrt{4\pi Dt}} \left(e^{-\frac{x^2}{4Dt}} - e^{-\frac{(x-2x_T)^2}{4Dt}} \right), \quad (78)$$

which allows to then determine $\bar{s}(t)$, $F(t)$, $G(t)$, and eventually

$$\bar{K}_r(\Delta) = fl_d \left[\frac{2}{\sqrt{\pi}} (1 - e^{-\eta^2}) + \eta \left(1 - 2\text{erf}(\eta) + \text{erf}\left(\frac{\eta}{2}\right) \right) \right], \quad (79)$$

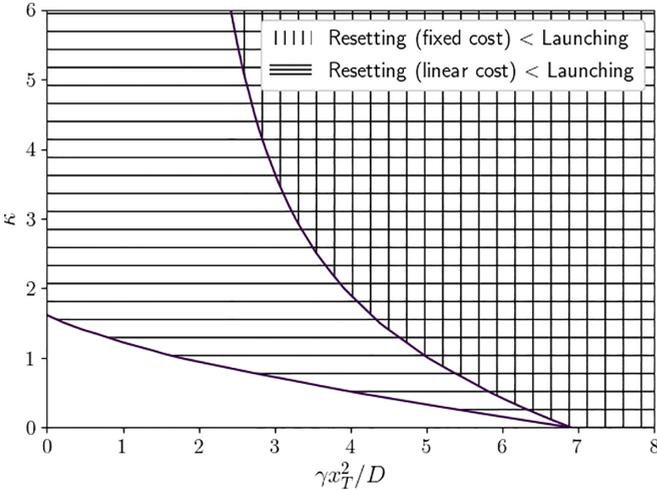


FIG. 7. Preferential strategy (resetting or launching new agents) for the one-dimensional diffusive search in the (κ, γ) plane, where we have used $\kappa = \kappa_r = f_r x_T$.

where we have defined $l_d = \sqrt{D\Delta}$ and $\eta = x_T/l_d$, which is equivalent to $f x_T$ as $\Delta \rightarrow \infty$. For a constant resetting cost, we simply have $\bar{\kappa}_r(\Delta) = \kappa_r^c$.

For both constant and linear resetting costs, we can minimize the total search cost \bar{K}_r with respect to Δ in order to find the optimal resetting time. We can then compare the corresponding optimal search cost with the optimal cost that one would find if one would launch new searchers at a constant time interval instead of resetting the first one. This optimal search cost in the launch process can be found by setting $t_i = (i - 1)\Delta$, calculating the corresponding search cost, and minimizing it with respect to Δ . Figure 7 shows for which values of κ and γ the optimal cost in the launch process is lower than the optimal search cost in the resetting process. To do this comparison, we need to set κ_r^c and f_r to a value which we choose as $\kappa = \kappa_r^c = f_r x_T$. It appears that launching new searchers is preferable to resetting the first one as long as the costs for sustaining a searcher, γ multiplied with the diffusive timescale, x_T^2/D , is sufficiently small, provided the costs for creating new searchers, κ , is also small enough. For linear resetting cost, there is even a value of κ above which resetting is always better than launching regardless of the value of γ .

Although we cannot draw general quantitative conclusions about the efficiency of launching compared to resetting from this single example since it is based on a specific model, it seems reasonable to assume that if κ and γ are not too large, the gain in the overall search time will dominate the costs of launching and sustaining new searchers. Note, however, that in practical applications, it could be much harder, if not unfeasible, to reset searchers, in contrast to adding new searchers.

VII. DISCUSSION AND CONCLUSION

In this paper, we have discussed the optimal strategies for when to launch noninteracting agents in a collective search process. We have defined a search cost that accounts for the overall search time, the amount of resources used by the searchers as they search and the cost for launching a

searcher. We have quantified when one should launch consecutive searchers in order to minimize the search costs, either its mean or its q quantiles—that is, the cost value for which the probability of observing lower costs is q^{-1} —and described it in terms of the single-agent survival probability of the target, i.e., the probability that the target has not been found after a certain time in a process with just one searcher. For both the mean and the quantiles, we have shown that if the SASP is logarithmically convex, there exists a unique local minimum of the search cost with respect to the launch times. Regarding the optimization of the quantiles, we have derived analytically the optimal launch strategy to minimize the probability for the search cost to be larger than a certain fixed value, which consists in launching an optimized number of agents initially and the next ones later than a certain time defined in terms of the SASP. We have also shown that this optimal strategy does not fix the launch times of later agents such that they can be adjusted to minimize subsequent quantiles. Regarding the mean search cost, we have shown that for situations where only one local minimum exists, the only instance where multiple searchers should be introduced simultaneously is at the start of the process and their optimal number is entirely governed by the launch cost and the initial slope of the SASP. Then, the details of the entire launching strategy were analyzed on a variety of test cases combining short- and long-time behaviors of the SASP. In particular, we have shown that the steeper the initial decay of the SASP is, the longer the time interval between the early searchers. However, the optimal time interval between launches of late searchers diverges if the SASP decays exponentially, while it reaches a constant value for algebraically decaying SASP. Finally, we have compared our results with search processes subject to stochastic resetting. Using the example of the one-dimensional diffusive search, we showed that launching searchers is preferable to resetting the first one to its initial position as long as the launch cost and sustaining cost are not too large.

Our work is the first to investigate in detail the optimization of a search process with respect to its number of searchers. The results exposed apply to a wide variety of processes but it must not be forgotten that some important assumptions were made along the way. In addition to the absence of interactions between agents, we also assumed that the SASP decays to zero, implying that each agent would eventually find the target if it would search for long enough. This is, e.g., not the case in diffusive searches in open space in dimension 2 or more, and it would be interesting to check to which extent the optimal strategies identified in this paper perform for such cases.

While the timing for the launches matters a lot for the overall search cost, as we have shown in this work, one might also wonder about the locations of the launches. Is there an optimal way to select initial positions of each searcher in order to minimize the search cost? This question is again reminiscent of stochastic resetting but in the context of collective searchers and should undoubtedly be treated in depth in the future. The analogy with recent studies on stochastic resetting can be stretched even further as one could also construct processes where the launching of new agents is conditioned on the evolution of external quantities, such as the proportion of the search domain that has been scanned at a certain time, or the current position of the agents already launched. This

type of questions enter the realm of decision making, a cross-disciplinary research field that can be used to substantially increase search efficiency and is a perfect playground for developing machine-learning algorithms. We believe that our work is a step in this direction and will motivate future studies to develop more and more refined collective search strategies.

ACKNOWLEDGMENTS

We acknowledge financial support by the DFG via the Collaborative Research Center SFB 1027. This work was supported by the Medical Research Council (Grant No. MR/Y003845/1).

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