Monte Carlo Studies of Ising Spin Glasses and Random Field Systems

Heiko Rieger

Institut für Theoretische Physik
Universität zu Köln,
D-50923 Köln, Germany

Abstract

We review recent numerical progress in the study of finite dimensional strongly disordered magnetic systems like spin glasses and random field systems. In particular we report in some details results for the critical properties and the non-equilibrium dynamics of Ising spin glasses. Furthermore we present an overview over recent investigations on the random field Ising model and finally of quantum spin glasses.

To appear in:
Annual Reviews of Computational Physics II
ed. D. Stauffer, World Scientific, Singapore (1995).

1 Introduction

Spin glasses and random field systems are magnetic materials in which a structural disorder occurs as a consequence of a special preparation process. The latter either changes the chemical composition of compounds and alloys (via dilution or mixing) or it decrystallizes the pure material (via sputtering or milling). The effect is in any case a randomness in the position of the spins and/or the sign and strength of the interactions among them. If this disorder is strong enough it can drastically change the magnetic properties of these material and can give rise to a new sort of phase transitions. In many cases even a completely new kind of low-temperature phase, the spin glass phase, might occur, which is in some sense ordered but does not possess a translationally invariant magnetic pattern. Theoretical models for these phenomena exist for over 20 years, however, it took about half of this time to establish the most fundamental facts for these models like the mere existence of a phase transition. The characterization of the latter transitions via the determination of the critical exponents is still far from settled. The investigation of the low temperature phase and its static and dynamic properties is a matter of ongoing experimental and theoretical research. A major role in these activities is taken by Monte Carlo simulations, which have found wide-spread applications in condensed matter physics [33].

Due to the enormous research activity on spin glasses many excellent reviews on this topic have already appeared during the last decade. To begin with, Binder and Young [31] and Chowdhury [61] give a complete survey on the spin glass literature until the year 1985. Also the proceedings of the two Heidelberg Colloquia on Spin Glasses [118, 119] contain useful references. Mézard et al. [181] focus on the mean-field theory of spin glasses and the book by Fischer and Hertz [87] is a rather complete introduction into this complex field. Mydosh's book [190] presents the most recent overview over experiments on spin glasses and a very stimulating collection of articles on random magnets [243] gives insight into ongoing experimental activities on spin glasses and random field systems. Excellent reviews of the latter have been given by Nattermann and Villain [192] focusing on theoretical aspects, by Belanger and Young [14] comparing theoretical predictions for the critical properties with the experimental situation and by Kleemann [152] about experimental evidence of the so called domain states, which is a characteristic feature of these systems.

Hence we do not intend to repeat the effort of the above reviews — we shall focus on frustrated Ising systems and its investigation via Monte Carlo simulations during recent years. In section 2 we review the status quo concerning the critical behavior of finite dimensional Ising spin glasses. Section 3 is devoted to the at present rapidly evolving field of the non-equilibrium, low-temperature dynamics of Ising spin glasses and other models exhibiting glassy features. Section 4 surveys our present knowledge of the critical properties and the off-equilibrium scenario of the random-field model and the diluted antiferromagnet in a uniform field. In section 5 we review the numerical methods that are useful in Monte Carlo studies of slowly relaxing systems like those considered here. Section 6 discusses quantum effects in spin glasses and especially the zero temperature quantum phase transition occuring in transverse-field Ising spin glasses. Finally section 7 lists other models like Heisenberg, Potts, orientational, vortex and Bose glasses.

2 Ising spin glasses: Critical properties

The simplest model that incorporates the necessary ingrediences for a spin glass as quenched disorder and frustration is the Edwards-Anderson model [83], which is defined by the Hamiltonian (or energy function)

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - h \sum_i S_i \ . \tag{1}$$

The spin variables are of Ising type, i.e. $S_i = \pm 1$, which are placed on a d-dimensional hyper-cubic lattice (simple cubic in three dimensions). The interactions among them are short ranged, which is indicated by the sum over only nearest neighbor pairs $\langle i, j \rangle$. The interaction strengths J_{ij} are quenched random variables that obey a given probability distribution, most commonly a Gaussian with mean zero and variance one or a binary distribution $J_{ij} = \pm 1$ with probability one half. The last sum in (1) describes the effect of an external magnetic field if $h \neq 0$. A typical experimental realization of this model is the short range Ising spin glass $Fe_{0.5}Mn_{0.5}TiO_3$ [136, 114] mixing ferromagnetically and antiferromagnetically interacting components.

A complete survey on the results for the critical properties of the two-, three- and four-dimensional Ising spin glass known until 1986 can be found in [23]. In essence the spin glass transition (at temperature T_c) in the infinite system is characterized by a divergence of the non-linear (and not the linear, as in e.g. ferromagnets) susceptibility

$$\chi_{SG} = \frac{1}{N} \left[\sum_{ij} \langle S_i S_j \rangle^2 \right]_{\rm av} \sim (T - T_c)^{-\gamma} \ , \tag{2}$$

which is caused by a divergence of the correlation length $\xi \sim (T - T_c)^{-\nu}$ that sets the characteristic length scale for the decay of spin correlations defined via

$$G(r) = [\langle S_i S_{i+\mathbf{r}} \rangle^2]_{\rm av} \sim r^{-(d-2+\eta)}\, \tilde{g}(r/\xi) \ . \tag{3}$$

Here $\langle \cdots \rangle$ means a thermal average and $[\cdots]_{\rm av}$ indicates the average over the quenched disorder.

Two different routes were used to check numerically the above scenario in three dimensions: One way is to perform Monte Carlo simulations of finite but large systems above T_c, where the correlation length ξ is smaller than the system size L, and to perform a scaling analysis of the spatial correlations $G(r)$. This has been done by Ogielski and Morgenstern [208] in extensive simulations on a special purpose computer. The other way is to confine oneself to smaller system sizes and to apply finite size scaling methods to analyze the data for the spin glass susceptibility χ_{SG} and certain combination of its moments, which has been done by Bhatt and Young [22, 24]. Both approaches come to the conclusion that there is indeed a finite temperature spin glass transition in the three-dimensional Ising spin glass model (1) with a binary distribution $J_{ij} = \pm 1$, the critical temperature is $T_c = 1.175 \pm 0.025$, the correlation length exponent is $\nu = 1.3 \pm 0.1$ and the exponent for the spin glass susceptibility is $\gamma = 2.9 \pm 0.3$. The results from high-temperature series expansions [270, 271, 294, 153] agree with these values within the errorbars.

At a second order phase transition the characteristic time scale τ for the decay of on-site correlations

$$q(t) = [\langle S_i(t) S_i(0) \rangle] \sim t^{-x} \tilde{q}(t/\tau) \quad \text{with} \quad x = (d-2+\eta)/2z \tag{4}$$

is also expected to diverge according to $\tau \sim \xi^z \sim (T - T_c)^{-z\nu}$ with z being the dynamical exponent. One possibility of defining an effective relaxation time is $\tau = (\int_0^\infty dt\, t\, q(t))/(\int_0^\infty dt\, q(t))$, for which Ogielski [209] finds $z = 6.1 \pm 0.3$ for the 3d $\pm J$ spin glass model. This is an unusually large value for z (which causes serious equilibration problems in Monte Carlo studies of the critical properties of this model) and an exponential divergence of τ at T_c like $\tau \sim \exp\{A/(T - T_c)\}$ cannot be excluded. Bernardi and Campbell [16, 17] perfomed a similar analysis for other bond distributions and found differences for the value of the exponent x depending on the particular form of these distributions, indicating that universality with respect to the latter possibly does not hold. Hukushima and Nemoto [127] investigated with the same method the two-dimensional case, where an exponential divergence of τ at $T = 0$ was found.

In a more recent work Blundell et al. [36] perform a scaling analysis of the non-equilibrium critical dynamics based on the hypothesis that the equal time spin-spin correlation function measured at time t after a temperature quench from $T = \infty$ to $T = T_c$ obeys

$$C(r,t) = [\langle S_i(t) S_{i+\mathbf{r}}(t) \rangle^2]_{\rm av} = r^{-(d-2+\eta)} f(r/t^{1/z}) \ . \tag{5}$$

Note that here $\langle \cdots \rangle$ means an expectation value with respect to the time dependent probability distribution generated by the underlying stochastic process and not the thermodynamic equilibrium expectation value. With other words, according to the hypothesis (5) one does not need to equilibrate the system to get estimates for the equilibrium critical exponents η and z. In this way a value $z = 5.85 \pm 0.3$ and $\eta = -0.29 \pm 0.07$ was obtained, concurring with the value reported above by Ogielski [209].

Another approach, suggested by Bhatt and Young [25], is to look at certain ratios of moments of time-dependent spin autocorrelation functions as for instance

$$R'(t) = \frac{[\langle N^{-1} \sum_i S_i(2t) S_i(t) \rangle]_{\rm av}}{\sqrt{[\langle N^{-1} \sum_i S_i(2t) S_i(t) \rangle^2]_{\rm av}}} \ . \tag{6}$$

Since this is a dimensionless quantity it is expected to scale like $R'(t) = \tilde{R}(t/\tau)$ with a characteristic time scale $\tau = \tau(L,T) \propto L^z \tilde{\tau}(L^{1/\nu}(T - T_c))$. Again the determination of equilibrium critical exponents proceeds via these hypothetical scaling forms of non-equilibrium quantities. Assuming that $T_c = 1.175$ (see above) in [25] it is found that $z = 6.0 \pm 0.5$, again in good agreement with Ogielski's result [209].

As we have seen so far there seems to be rather compelling evidence that the three-dimensional Ising spin glass in vanishing external field has a second order phase transition at a finite temperature T_c. However, a recent Monte Carlo study by Marinari et al. [172] of the three-dimensional Ising spin glass on a body-centered cubic lattice, where second and third nearest neighbor interactions have been taken into account in order two increase the hypothetical critical temperature, revealed a less clear picture: It was found that in this model the data are compatible with a non-zero temperature phase transition (with $T_c = 3.25 \pm 0.05$, which is a non-universal quantity and therefore different from the value reported above, $\nu = 1.20 \pm 0.04$ and $\gamma = 2.43 \pm 0.05$) as well as with a zero temperature transition:

$$\begin{aligned} \chi_{SG} &\sim A[\exp(B/T)^4 - 1] + C \ , \\ \xi &\sim a[\exp(b/T)^4 - 1] + c \ . \end{aligned} \tag{7}$$

Marinari et al. [172] report that this form fits their data better with a smaller systematic error, thus giving new support for the speculation that three dimensions may well be (instead of close to but larger than) the lower critical dimension of the short ranged Ising spin glass model.

In any other number of dimensions the situation is much clearer: In two dimensions it was shown that no phase transition is present at any finite temperature [217, 179, 60, 300, 180, 45, 128, 283, 24]. For a ± 1 distribution the most recent results are from Bhatt and Young [24] reporting that $T_c = 0$, $\nu = 2.6 \pm 0.4$ ($\xi \sim T^{-\nu}$) and $\gamma = 4.6 \pm 0.5$ ($\chi_{SG} \sim T^{-\gamma}$). A continuous distribution yields significantly different results indicating that this case is in a different universality class than the ± 1 distribution, since the latter has a large ground state degeneracy. In four dimensions one finds much stronger indications for the existence of a phase transition than in three dimensions [24]: For a Gaussian distribution it is found that $T_c = 1.75 \pm 0.05$, $\nu = 0.8 \pm 0.15$, $\gamma = 1.8 \pm 0.4$ [24] and $z = 4.8 \pm 0.4$ [25]. Furthermore the extensive investigation of the probability distribution $P(q)$ of the replica overlap q [226, 220, 63] revealed a picture that is reminiscent of the Sherrington-Kirkpatrick (SK) model [262] and its solution with broken replica symmetry [181]. The upper critical dimension of the Ising spin glass, above which mean field results should apply, is expected to be six [117]. This prediction has been confirmed by Monte Carlo simulations of the six-dimensional Ising spin glass with ± 1 couplings [295], where at $T_c = 3.03 \pm 0.01$ mean field critical behavior is found, which is $\nu = 1/2$ and $\gamma = 1$, with logarithmic corrections due to being *at* the upper critical dimension.

The effect of an external magnetic field on the critical behavior of the Ising spin glass is also far from clarified. The mean field scenario [181] would suggest the existence of an AT (de Almeida-Thouless [3]) line in the h-T–phase diagram and thus a spin glass transition even within a non-vanishing external field. On the other hand phenomenological models [89, 90] dispute the relevance of mean-field results for finite dimensional spin glasses and predict the absence of a phase transition in a field. All numerical results obtained so far for four dimensions [110, 8, 64, 240] seem to be compatible with the existence of a spin glass transition at a nonvanishing temperature in a low enough external field.

In three dimensions the situation is still unclear: Kawashima et al. [144, 145] have investigated the variance of the probability distribution of the central spin magnetization as well as the probability distribution of the replica overlap and observe that for increasing system size it scales to zero in non-vanishing external fields. This implies the non-existence of a phase transition. Following a proposition by Singh and Huse [272], Grannan and Hetzel [110] looked at the contour lines $\chi_{SG}(T, h) = 1$ in the T-h diagram and observe an upwards bend (i.e. $h \to \infty$) for $T \to 0$ in four dimensions (which indicates a spin glass phase) and a downwards bend (i.e. $h \to 0$) in two dimensions (meaning the absence of a spin glass phase). This is depicted in figure 1. The three-dimensional case is only marginally bending upward, suggesting that three is close to but larger than the lower critical dimension also in an external field (see also [120] for the same study of the 3d Ising spin glass on the face-centered cubic lattice). Ritort [240] came to simlar conclusions by investigating the quantities describing the sensitivity of the spin glass phase with respect to magnetic-field perturbations. Finally Caracciolo et al. [51, 52, 54] presented results of Monte Carlo simulations of the (three-dimensional) system

$$H = -\sum_{\langle i,j \rangle} J_{ij}(\sigma_i\sigma_j + \tau_i\tau_j) + \sum_i h(\sigma_i + \tau_i) + \varepsilon \sum_i \sigma_i\tau_i \ , \tag{8}$$

which is identical to the two-replica method used by Bhatt and Young [22] apart from the existence of a coupling of strength ε between the two replicated systems σ and τ. This parameter ε plays a similar role as the magnetic field in a ferromagnet — it is the conjugate field to the order parameter q, the replica overlap. Hence one expects a discontinuity in $q(\varepsilon)$ in the limit $\varepsilon \to 0$ for temperatures $T < T_c$, if there is a phase transition. The authors conclude that their results are inconsistent with the predictions of the droplet model [88] and support the mean field picture including replica symmetry breaking and the existence of an AT-line (i.e. a spin glass transition within a field) even in three dimensions. These claims have been heavily disputed by Huse and Fisher [132] (see also the subsequent reply by Caracciolo et al. [53]), the main counter-argument being that their simulations are performed at temperatures *above* the hypothetical AT-line and that the system sizes are much too small to give reliable results for the low field regime. We will return to the issue of the correctness of the droplet theory in finite dimensional spin glasses below.

3 Ising spin glasses: Non-equilibrium dynamics

Most experiments on spin glasses at low temperatures are performed in a non-equilibrium situation due to the astronomically large equilibration times. Since the seminal work of Lundgren et al. [169] it has been realized by the experimentalists [170, 126] that magnetic properties of spin glasses strongly depend on the time they spent below the spin glass transition temperature T_c, a phenomenon that has been dubbed *aging*. (see e.g. [164, 292] for a review about the experimental situation). Figure 2 shows the result of a typical aging experiment: The spin glass is rapidly cooled to low temperatures in an external field h (in three dimensional spin glasses below the transition temperature) and the field is switched off only after a macroscopic (several hours) waiting time t_w. As soon as the field is zero (which defines the time $t = 0$) the time-(t)-dependence of the thermoremanent magnetization $M_{\rm TRM}(t, t_w)$ is measured. Usually (as for instance in simple ferromagnets) the equilibration time is microscopic and for macroscopic waiting times one would measure always the same magnetization curve $M_{\rm TRM}(t, t_w)$ independent of t_w. As can be seen in figure 2 this is completely different in spin glasses, which is an obvious manifestation of the huge time scales of the glassy dynamics. In fact a spin glass transition is not a necessary ingredient for this scenario to occur, it has also been reported in experiments with two-dimensional spin glasses like $Ru_2Cu_{0.89}Co_{0.11}F_4$ [78, 79] or CuMn-films [175] and in models without frustration [238] or disorder [150, 73]. The observation characteristic for aging can also be made in other amorphous or strongly disordered materials like polymers [280], charge-density-wave systems [26, 27] or high-temperature superconductors [241].

In Monte Carlo simulations the spin autocorrelation function

$$C(t, t_w) = \frac{1}{N} \sum_{i=1}^{N} [\langle S_i(t + t_w) S_i(t_w) \rangle]_{\rm av} \tag{9}$$

is the quantity that is most convenient to investigate the non-equilibrium dynamics of spin systems. Here, as in equation (5), $\langle \cdots \rangle$ formally means an average over various thermal histories and initial conditions. However, as long as the number of samples for the disorder average $[\cdots]_{\rm av}$ is large enough, it does not make a difference (numerically) if one neglects this thermal average.

The limit $t_w \to 0$, which corresponds to experimental measurements of the remanent magnetization $M_{\rm TRM}(t)$ after removing a strong external field (of the order of the saturation field strength), measures the configurational overlap with a fully magnetized initial state. Early investigators of this quantity [28, 146, 147] observed already an algebraic decay in a certain time window at low temperatures. This has been confirmed recently for the SK-model [84, 218] and finite dimensional spin glass models [129, 85, 218]. In more extensive simulations for the 3d EA-model [234] the temperature dependent exponent $\lambda(T)$ in

$$M_{\rm TRM}(t) \sim t^{-\lambda(T)} \ , \tag{10}$$

has been determined for various temperatures below the spin glass transition temperature and found a good agreement with those reported for the amorphous metallic spin glass $(Fe_xNi_{(1-x)})_{75}P_{16}B_6Al_3$ [107], which is particularly well suited for measurements of the remanent magnetization also for low temperature since its saturation field strength is reasonably low (see figure 3).

For non-vanishing waiting times $t_w \neq 0$ the function (9) shows a behavior that is characteristic for aging phenomena. Andersson et al. [4] and Rieger [234] showed that qualitative features of experimental observations can be reproduced by Monte Carlo simulations of finite dimensional spin glass models. In addition to $C(t, t_w)$ the magnetic response function $\chi(t, t_w) = M_{\rm TRM}(t, t_w)/h$ has been determined, where $M_{\rm TRM}(t, t_w)$ is defined in the introductory paragraph of this section. Its logarithmic derivative $\partial\chi(t, t_w)/\partial \log t$ possesses a maximum at $t = t_w$, as observed in the corresponding experiments [199, 108]. Furthermore one observes that the fluctuation–dissipation theorem $\chi(t, t_w) = \beta(1 - C(t, t_w))$ is violated for $t > t_w$. A more quantitative analysis of the the data for the function $C(t, t_w)$, of which a typical set is depicted in figure 4, reveals the following picture [234]: In three dimensions there are strong indications for the scaling relation

$$C(t, t_w) = t^{-x(T)}\Phi_T(t/t_w) \quad \text{with} \quad \Phi_T(y) \sim \begin{cases} \text{const.} & \text{for} \quad y \to 0 \,, \\ y^{x(T)-\lambda(T)} & \text{for} \quad y \to \infty \end{cases} \tag{11}$$

to hold This characteristic t/t_w-scaling has also been observed in experiments with the insulating spin glass $CdCr_{1.7}In_{0.3}S_4$ [206, 2], in two-dimensional spin glasses [237], the SK-model [71], in simplified spin glass models [40, 219] and one-dimensional models [238]. For a spin glass below the critical temperature the relation (11) is expected to hold for any finite waiting time t_w (as long as the system is infinite), which expresses the fact that for all temperatures below T_c the equilibration time is infinite — in contrast to the situation in e.g. simple ferromagnets. For $t_w \to \infty$ one obtains the equilibrium autocorrelation function $\lim_{t_w\to\infty} C(t, t_w) = q(t) \sim t^{-x(T)}$, see equation (4). It should be noted that in spin glasses with replica symmetry breaking a more complicated scenario can occur, see [97] for a discussion. At T_c one has $x(T_c) \approx 0.07$, which equals the exponent reported for the equilibrium dynamics of the 3d EA-model [209] as well as from experiments on the short-range Ising spin glass $Fe_{0.5}Mn_{0.5}TiO_3$ [114]. In other words: as long as $t \ll t_w$ one observes a quasi-equilibrium dynamics (on time scales smaller than the waiting time t_w) and the crossover at $t = t_w$ signals the onset of the non-equilibrium behavior characterized by an exponent $\lambda(T)$ that is significantly larger than the corresponding equilibrium exponent $x(T_c)$ (e.g. $\lambda(T_c) \approx 0.39$, for this value see also [129]).

At this point a short excursion on semantics might be appropriate. **A:** When speaking of equilibration time in the preceeding paragraph we mean the equilibration time within one ergodic component of an infinite system, like the one with positive (or negative) expectation value for the magnetization in a ferromagnet below T_c. This definition is independent of the answer to the question, whether the phase space of a finite dimensional spin glass splits into an infinite number of ergodic components separated by infinite free-energy barriers below T_c (as the SK-model) or not. We observe that $\lim_{t\to\infty} C(t,t_w) \to 0$ for any waiting time t_w, but the more natural expectation for dynamics within an ergodic component would be an exponential decay of $q(t)$ to a finite value q_{EA}. However, no indication for this has been reported yet [207, 234, 71].

B: The scenario (11) is what we call *aging*, and we use the expression *interrupted aging* for a situation in which we have some finite equilibration time $\tau_{\rm eq}$ in such a way that (11) holds approximately as long as $t_w \ll \tau_{\rm eq}$ and is simply replaced by true equilibrium dynamics $C(t,t_w) \approx q(t)$ for $t_w \gg \tau_{\rm eq}$. This is what happens in two-dimesnional spin glasses [247, 248, 237] or random ferromagnets [238], but also in pure systems like the one considered in [150], where depending on the system parameters the time $\tau_{\rm eq}$ can be extremely large. Obviously for an astronomically large $\tau_{\rm eq}$ neither experiments nor Monte Carlo simulations might be able to discriminate between aging and interrupted aging. It has also been noted [150] that in the pure ferromagnetic Ising chain the aging scenario (11) holds at $T = 0$ with $x = 0$, thus aging phenomena do not rely upon randomness or frustration althought the latter can greatly enhance it. We would like to stick to this nomenclature in this review. However, there exists also a different perception of what aging might be [70, 72, 183, 96], see also the discussion following eq. (16) below. In essence, aging understood in this way is a property of off-equilibrium dynamics that becomes manifest only in particularly chosen limiting procedure for the times t and t_w. A scenario like (11) with $x(T) > 0$ would be called "interrupted aging", even if $\tau_{\rm eq} = \infty$. This concept of aging is motivated by analytically tractable mean-field models, where various infinite time limits can be done in a straightforward manner — its relevance for real spin glasses and finite time experiments remains to be tested.

The underlying mechanism for the aging scenario (11) in finite dimensional spin glasses is a slow domain growth, as suggested by Fisher and Huse [91] proposing (ad hoc) a logarithmic growth law for the characteristic domain size $R(t) \propto (\log t)^{1/\psi}$, and by Koper and Hilhorst [157], stipulating (ad hoc) an algebraic growth $R(t) \propto t^{\alpha(T)}$. Preliminary investigations on this issue [131] find some numerical support for the first hypothesis, however, the above mentioned facts (the asymptotically algebraic decay of $C(t,t_w)$ and the t/t_w scaling) and recent very extensive simulations pledge more in favor of an algebraic growth: Rieger et al. [237] perform Monte Carlo simulations of the two-dimensional EA-model with Gaussian couplings and investigate the growth of spatial correlations via the correlation function

$$G(r,t_w) = \frac{1}{N}\sum_{i=1}^{N}[\langle S_i(t_w)S_{i+\mathbf{r}}(t_w)\rangle^2]_{\rm av} \ , \tag{12}$$

which becomes $G(r)$ of equation (3) in the limit $t_w \to \infty$. Note that the domain growth in various strongly disordered systems like the site-diluted Ising model, the random field Ising model and the random bond ferromagnetic Ising model, has been investigated frequently via Monte Carlo simulations (see the review by Chowdhury and Biswal in this series [62]). These models have the advantage that their ground state is known to be ferromagnetic, which makes the identification of domains easy. However, this is not the case for the present system and such an investigation is much more difficult here. Hence, a straightforward way to determine a typical length scale for spatial correlations in a spin glass is to calculate the function $G(r,t_w)$ defined in (12).

In order to take into account the square of the thermal average in (12) in [237] two replicas a and b of the system were simulated. Then $[\langle S_i^a(t_w)S_i^b(t_w)S_{i+\mathbf{r}}^a(t_w)S_{i+\mathbf{r}}^b(t_w)\rangle]_{\rm av}$ instead of $[\langle S_i(t_w)S_{i+\mathbf{r}}\rangle^2]_{\rm av}$ was calculated, giving the same results with a much better statistics for the first quantity. To improve the statistics one also has to average over a suitably chosen time window around t_w. The correlation length is defined via $\xi(t_w) = 2\int_0^\infty dr G(r,t_w)$. It turns out (see figure 5) that

$$\xi(t_w) \propto t^{\alpha(T)} \ , \tag{13}$$

with an exponent $\alpha(T)$ that decreases with temperature yields a good fit to the time dependency of the correlation length. Of course in the two-dimensional spin glass, which does not have a phase transition at any finite temperature (see previous section), $\xi(t_w)$ has to saturate at a finite value. This, however, happens at small temperatures ($T \sim 0.2$) only at very large times t_w inaccessible to computer simulations. It should be noted that a logarithmic fit $\xi(t_w) \sim (\log t)^{1/\psi}$ also works fairly well — with a temperature independent exponent $\psi \approx 0.63$. In recent experiments on the two-dimensional spin glass $Ru_2Cu_{0.89}Co_{0.11}F_4$ [247, 248] the scaling behavior of the ac-susceptibility has been analyzed and it turned out that this is compatible with a prediction made by Fisher and Huse [91] assuming a logarithmic growth with $\psi \approx 1.0$. Unfortunately no direct measurements of the correlation length itself are available experimentally up to now.

For three dimensions the same kind of calculations as described above have been done [151] (see also the work by Sibani and Andersson [268, 6] discussed below) which indicate the validity of algebraic growth law (13), too. A continuous set of exponents like $\alpha(T)$ for domain growth (or sometimes called coarsening) might be disturbing especially if one considers the work of Lai, Mazenko and Ma [163]. They classify various non-equilibrium systems according to the underlying scaling law for (free) energy barriers $B(L)$ that domains of size $L(t)$ have to overcome in order to grow further: two of them lead to $L(t) \propto \sqrt{t}$ and two lead to logarithmic growth — the fourth being identical to the proposal of Fisher and Huse [91] for spin glasses $B(L) \sim L^{\psi}$, which leads to $L(t) \sim (\log t)^{1/\psi}$ via activated dynamics. However, as suggested by Rieger [234], the scaling law

$$B(L) = \Delta(T) \log L \tag{14}$$

leads (via an activated dynamics scenario in which it takes a time $\tau \sim \exp(B/T)$ to overcome a (free) energy barrier B) to an algebraic domain growth with a continuous set of exponents as in (13). Moreover, following [91] in their argumentation, we stipulate for the remanent magnetization that $M_{\rm TRM}(t) \sim L(t)^{-\delta}$ and for the autocorrelation-function that $C(t, t_w) \sim [L(t_w)/L(t + t_w)]^{\delta}$. Then it follows $M_{\rm TRM}(t) \sim t^{-\lambda(T)}$, $C(t, t_w) \sim t^{-\lambda(T)}$ for $t \gg t_w$ and also the t/t_w scaling stated in (11). The following table gives a collection of predictions for the different scaling assumptions.

		Droplet-model [91]	MC-sim.	
Energy barrier	$\mathbf{B}$	ΔL^{ψ}	$\Delta(T) \log L$	$\bullet$
Activated dynamics	τ	$\tau \sim \exp B/T$	$\tau \sim \exp B/T$	
Domain size	$\mathbf{R(t)}$	$\left(\frac{T}{\Delta} \log t\right)^{1/\psi}$	$t^{\alpha(T)}$	$(\bullet)$
Remanent Magnetization	$\mathbf{M}_{TRM}$	$\left(\frac{T}{\Delta} \log t\right)^{-\delta/\psi}$	$t^{-\lambda(T)}$	$\bullet$
Aging	$\mathbf{C(t,t}_w\mathbf{)}$	$\overline{C}\left(\frac{\log t+t_w}{\log t_w}\right)$	$\tilde{C}\left(\frac{t}{t_w}\right)$	$\bullet$
Asymptotic decay	$\mathbf{t} \gg \mathbf{t}_w$	$(\log t)^{-\delta/\psi}$	$t^{-\lambda(T)}$	$\bullet$
	$\mathbf{t} \ll \mathbf{t}_w$	$(\log t)^{-\theta/\psi}$	$t^{-x(T)}$	$(\bullet)$

The bullets indicate that the Monte Carlo simulations described above and below confirm the corresponding prediction in the last colomn. A bullet with brackets means that the numerical data can also be interpreted according to the predictions of the droplet model. Concluding the hypothesis (14) seems to give a more consistent description of experimental and numerical results on aging in spin glasses presented so far.

Recently Sibani and Andersson [268, 6] gave further support for the relation

(14) by means of the following procedure: In Monte Carlo simulations of the two– and three–dimensional spin glass reference states Ψ are generated with the help of a careful annealing procedure down to $T = 0$. Then a certain amount of spins is flipped randomly under the constraint that the energy difference of the new state to the old reference state remains below a certain lid b. Via zero-temperature dynamics the system has the opportunity to relax into a local energy minimum configuration that is stable against single spin flips. Finally the new state is analyzed by identifying all clusters of reversed spins. For these clusters the size and energy distribution is recorded. It turns out that the cluster volume increases exponentially (or even superexponentially) with the lid b, implying a logarithmic dependence for the energy barriers as in (14). This dependency disagrees with the result of Gawron et al. [100], who use an exact search algorithm for calculating the energy barriers against inversion of ground states. They obtain in two dimensions $B(L) \sim L$, which means $\psi = 1$. A serious caveat in their approach might be the fact that their investigations have been constrained to system sizes $L \leq 5$.

Especially with regards to the latter investigations it seems that the systematic analysis of the phase-space structure of short range spin glasses, especially their ground states and low lying excitation, seems to become feasible with increasing computer power These studies already gave valuable insights [266, 267, 154, 155], supporting the phenomenological theories of hierarchical relaxation for the spin glass dynamics á la Schreckenberg [249], Ogielski and Stein [207], which have been improved further by Sibani and Hoffmann [265, 251, 124] in order to model simple aging, temperature step experiments and violation of the fluctuation dissipation theorem. In particular with regards to improved optimization algorithms [111, 10, 281, 269] (see also [143] and the articles by Stariolo and Tsallis [276], Hogg [125] and Tomassini [288] in this series) significant progress in the investigation of ground state properties in spin glasses can be expected in the future.

Another procedure devised to test various phenomenological spin glass theories are so-called temperature cycling experiments. They consist of two temperature changes during the time in which the material is aged in the spin glass phase [225]: either a short heat pulse is applied to the spin glass during the waiting time after which the relaxation of e.g. the thermo-remanent magnetization is measured, or a short negative temperature cycle is performed, which is the same as a heat pulse but with a negative temperature shift during the pulse. It has been pointed out [166] that this kind of experiments can discriminate between the droplet picture [91] and the hierarchical picture [164] (see also [134] and [99] for a microscopic theorie of adiabatic cooling, which are also capable to explain some of the obsevations). The interpretation of the experimental situation is still controversial (see also [296] for a refreshing description of the present situation) — essentially the Saclay group pledging in favor of a hierarchical interpretation [225, 166, 292] and the Uppsala group taking the part of the droplet model [109, 175]. The situation is the same with regards to numerical simulations of temperature cycling experiments: Rieger [235] comes to similar conclusions as the former and Andersson et al. [5] interpret their results in full agreement with the latter experiments. In both Monte Carlo simulations the same quantities are studied (for instance the thermoremanent magnetization $M_{\rm TRM}(t,t_{age})$, where t_{age} now means the whole time in which the temperature cycle has been performed), but the resulting data are analyzed differently. An alternative approach to the study of the temperature dependence of the spin glass state has been suggested by Jan and Ray [140] using damage spreading (for the use of the latter concept in Ising spin glasses see [80, 7, 57] and the article by Jan and de Arcangelis in this series [141]).

It might be difficult to make conclusive progress in this direction, because these kind of experiments (real and numerical) always yield ambiguous results that invite to one or the other interpretation. However, the predictions of the droplet theory heavily rely on the concept of *chaos* in spin glasses [47, 197, 198] (meaning the significant sensibility of the spin glass state to infinitesimal changes of parameters like temperature or field) that can be quantified by an overlap length $\zeta(T,\Delta T)$ via the hypothesis

$$[\langle S_i S_{i+r}\rangle_T \langle S_i S_{i+r}\rangle_{T\pm\Delta T}]_{\rm av} \sim \exp\{-r/\zeta(T,\pm\Delta T)\}\ , \qquad (15)$$

where $\langle\cdots\rangle_T$ means the equilibrium expectation value of one system at temperature T. The characteristic length scale $\zeta(T,\pm\Delta T)$ should be finite for the positive and negative sign of ΔT and it should decrease with increasing ΔT. There are quantitative predictions for these dependencies, inaccessible to experiments, which can, however, be calculated in Monte Carlo simulations. This seems to be a promising endeavor. It should be noted that the existence of an overlap-length as defined in (15) is a prediction of phenomenological concepts [47, 90, 91] *and* microscopic theories on the basis of the SK-model, as has been reported by Kondor [156] and discussed further by Ritort [240].

It should be mentioned that very recently very appealing theoretical concepts for the non-equilibrium dynamics in spin glasses have been developed. The phenomenological theory that fits the experimental data for aging experiments in the best way up to now was introduced by Bouchaud [40, 41, 43]. In essence it is a diffusion model in an abstract space in which each state is characterized by a random (free) energy and hence by a random, exponentially distributed trapping time. By arranging them in a tree like structure, very reminiscent to the phase space structure that emerges from Parisi's solution of the SK-model [181], one obtains functional forms for $M_{TRM}(t, t_w)$ and $C(t, t_w)$ that yield excellent fits to experimentally measured data. Schreckenberg and Rieger [250] propose a different diffusion model, which is based on an ultrametric tree that incorporates a separation between quasi-equilibrium and non-equilibrium branches. In this way aging occurs naturally via exploration of quasi-equilibrium sub-branches of increasing depth. Also the microscopic theory for the off-equilibrium dynamics of mean field models of spin glasses has been pushed forward by Cugliandolo and Kurchan [70, 72] and Franz and Mézard [183, 96]: The mathematical difficulties for an analytically exact solution of the dynamical off-equilibrium mean field equations for e.g. the SK-model come from the lack of the fluctuation-dissipation theorem (FDT) that relates autocorrelation and response function (which allows the analytical solution in case of equilibrium dynamics [274, 273, 68, 148]). The new approaches circumvent this by considering a so-called fluctuation-dissipation ratio defined via

$$x(t,t') = \frac{r(t,t')}{\beta \partial C(t,t')/\partial t'}. \tag{16}$$

and postulating a particular set of properties for this function $x(t, t')$ in various asymptotic limits, essentially setting up an "ultrametric" for timescales. In this way Parisi's static, equilibrium (!) order parameter function $q(x)$ finds its counterpart in off-equilibrium dynamics. It will be a challenging endeavor to test these interesting ideas via Monte Carlo simulations and to check, whether they might also be applicable in finite dimensions. The analysis of Monte Carlo results for the three-dimensional EA spin glass are compatible with the proposed scenario [97] although, due to the marginality of the three-dimensional case, further investigation in higher dimensions are desirable and would certainly give a much clearer picture.

Finally we would like to point out that aging and glassy dynamics have gained much interest very recently. Apart from the issues mentioned above in connection with spin glasses the central point of the research activities is to model glassy behavior with spin systems that have no quenched disorder (in order to try to understand the glass transition that leads for instance to the formation of window glass): among them are two- and three-dimensional models with competing nearest and next-nearest neighbor interactions [259, 264], the anisotropic kagomé antiferromagnet [229, 58], one-dimensional spin models with p-spin interactions [150], the Bernasoni model (also being a p-spin interaction model, but with infinite range interactions) [42, 173, 185, 159] and certain field theoretial models with infinite range interactions [74, 98]. Apart from the latter two, the bulk of investigations of this very fascinating subject has to be done numerically via Monte Carlo simulations.

4 Numerical recipes

It is obvious that thorough Monte Carlo simulations of spin glasses require a huge amount of computer time. As long as one is interested in purely dynamical phenomena as described in the previous section, there is no other way than to implement some algorithm that generates a stochastic process described by the Master equation for the probability distribution of the spins. The possibility that is most frequently used to achieve this, is the heat bath, Glauber or Metropolis algorithm. Here a random number generator is used to decide, whether a certain spin configuration $\underline{S}$ is modified or not, according to predefined transition probabilities, as for instance

$$w(\underline{S} \to \underline{S}') = \min\left\{1, \exp(-\beta\Delta E)\right\} , \qquad (17)$$

where ΔE is the energy difference between the old ($\underline{S}$) and new ($\underline{S}'$) configuration. For Ising systems very efficient implementations of this algorithm with single spin flips have been presented that reach a speed of $3 \cdot 10^8$ spin update attempts per second on a single processor of a Cray YMP: for the pure Ising model by Ito and Kanada [137] and Heuer [121], for the random field Ising model [231] and finally for a whole class of Ising models with or without quenched disorder defined via binary variables [138] (see also the book of de Olivera [213] for a review on multi-spin coding techniques). The enormous speed up compared to older multi-spin coding techniques is based upon the use of a single random number for different (up to 64) systems, like e.g. 64 different samples of random bond configurations of the EA spin glass. This method does not work for a continuous probability distribution of the quenched disorder variable (like Gaussian) — in this case more conventional methods have to be used, which are at least one order of magnitude slower. Although nearly all results reviewed above have been obtained with these more or less optimized algorithms they reveal serious deficiencies at low temperatures:

Equation (17) means that the new configuration $\underline{S}'$ is rejected if a random number, equally distributed between 0 and 1, is larger than $w(\underline{S} \to \underline{S}')$, therefore this is called a rejection method. In system with a huge number of local energy minimum configurations the number of rejections becomes very large at low temperatures and most of the CPU time is wasted by generating random numbers and calculating (or looking up in a table) the transition rates. The only effect is to increment the time that has passed since the beginning of the simulation by one unit. Bortz, Kalos and Lebowitz [39, 29] suggested already 20 years ago a method that seems to be more efficient: They proposed to accept a new configuration *always* and then to increment the time by a random number Δt obeying a probability distribution that is characterized by the sum over all probabilities for transitions away from the old configuration:

$$P(\Delta t) = \tau^{-1} \exp(-\Delta t/\tau) \quad \text{with} \quad \tau^{-1} = \sum_{\underline{S}'} w(\underline{S}' \to \underline{S}') \tag{18}$$

Although this idea might be capable of circumventing the above mentioned slowing down it has not been used very frequently up to now [103, 264, 85, 160]. One of the problems that typically occur also here is that once a transition away from a local minimum configuration has taken place the system will very soon return to it and thus decrementing the efficiency of the algorithm. Very recently Krauth and Pulchery [159] proposed a variant of this method that seems even to avoid this caveat of short cycles: By keeping track of the configurations already visited (which means storing them into the computer memory) the algorithm is forced to generate new configurations in each iteration. The memory is cleared as soon as a lower local energy minimum is encountered. In this way they were able to explore times scales equivalent to 10^{13} conventional Monte Carlo steps for a particular spin model with 400 spins. This is very promising and would mean a real breakthrough if such a performance could also be achieved with finite dimensional spin glass models of reasonable size.

A completely different method that tries to avoid the tremendous slowing down caused by coexisting energy minima in phase space with large energy barriers between them is the so-called multicanonical ensemble [18] or simulated tempering [171], which is suited for the calculation of equilibrium properties and also ground states of spin glasses. Usually (as in the above mentioned algorithms) one performs an importance sampling with the canonical distribution $P_{\rm can.}(E) \sim \exp(-E/T)$, which is sharply peaked around one (at high temperatures T), two (in case of first order phase transitions) or several (in spin glasses or random field models) values of the energy E. In order to escape from one energy minimum region to explore others and, in particular for interface problems, the regions in between them, one tries to generate an ensemble $P_{\rm m}(E)$ that is approximately flat for the energy interval of interest, especially between the sharp maxima of the canonical distribution. This can be achieved with appropriately chosen functions $\alpha(E)$ and $\beta(E)$ in the multicanonical ensemble

$$P_{\rm m}(E) \sim \exp[-\beta(E)E + \alpha(E)] \; . \tag{19}$$

Since they are unknown a priori, they have to be determined recursively during the simulations [18, 19]. Finally, by re-weighting with $\exp[-E/T + \beta(E)E - \alpha(E)]$, the canonical distribution can be reconstructed. With regards to spin glasses, this ensemble has proven to be useful in particular for the investigation of ground state properties [19, 20, 21] (for this problem see also the above mentioned works [111, 10, 281, 269, 276, 125, 288]).

Cluster algorithms as suggested by Swendsen and Wang [284] have become very useful in the investigation of pure systems (see [285] for an overview). However, they do not yet give a significant improvement over local algorithms simulating single spin flip dynamics in spin glass or random field models — with at least one exception: In a special implementation of a cluster algorithm for the two-dimensional Gaussian EA-model Liang showed [167] that the logarithm of the relaxation time is five times smaller than the usual Metropolis algorithm. But he also pointed out that this efficiency will not be reached in higher dimensions. A cluster algorithm for the random field Ising model has also been proposed [81], but its efficiency remains to be tested. The reason for this is that due to randomness and frustration it is not obvious at all how to construct spin clusters that can be reversed with acceptable rates. Since the correct construction of these clusters is the main problem in spin glasses we should also mention the work on various cluster concepts in disordered systems [277, 224, 279, 112].

Finally let us mention histogram techniques [86] that also have been used frequently for pure systems [285]: in principle they allow to get information about thermodynamic quantities of one system (i.e. one realization of the quenched disorder) in a whole temperature interval by a Monte Carlo run at one single temperature. However, this needs a lot of book keeping already for pure systems, where several thousand configurations have to be stored. In spin glasses and random field systems in addition one has to perform an average also over this disorder, which might cause a serious difficulty for data storage.

5 Random field systems

If one mixes a typical antiferromagnet like $FeCl_2$ with an isostructural nonmagnetic material like $CoCl_2$ or $NiCl_2$ one obtains diluted antiferromagnets, which are discussed in this series [62, 257]. Within an uniform external field, however, a large degree of frustration is induced and a completely new universality class (with regards to critical properties) emerges. Fishman and Aharony [95] and also Cardy [55] pointed out that it is the same as the universality class of the random field Ising model (RFIM), which is defined by the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i \ , \tag{20}$$

where the first sum is again over nearest neighbor pairs on a d-dimensional lattice and the h_i are independent quenched random variables with $[h_i]_{\rm av} = 0$ and $[h_i^2]_{\rm av} = h^2$. A simple heuristic argument by Imry and Ma [133] shows that the lower critical dimension of the RFIM (20) should be $d = 2$. Indeed it has been proved rigorously [48] that in three dimensions for small enough random field strength h there is ferromagnetic long range order at low enough temperatures. Thus the existence of a phase transition in three (and higher) dimensions is assured (in so far the situation is slightly better than in spin glasses), however, there has been a long lasting debate on the critical properties, which still awaits a solution.

Villain [291] and Fisher [88] proposed a scaling theory for the random field transition that relies upon the assumption that random field induced fluctuations dominate over thermal fluctuations at T_c. This implies for the singular part of the free energy

$$F_{\rm sing} \sim \xi^\theta \ , \tag{21}$$

where ξ is the correlation length and θ is a new exponent. Random field fluctuations alone produce typically an excess field of the order of $\xi^{d/2}$ within a correlation volume, so a naive guess would be θ roughly 1.5 in three dimensions. From (21) the modified hyperscaling relation follows

$$2 - \alpha = \nu(d - \theta) \ , \tag{22}$$

and it also implies an exponential divergence of the relaxation time τ at T_c: $\tau \sim \exp(A/\xi^\theta)$. The decay of the connected ($[\langle S_0 S_r \rangle]_{\rm av} - [\langle S_0 \rangle \langle S_r \rangle]_{\rm av} \sim r^{-(d-2+\eta)}$) and disconnected ($[\langle S_0 \rangle \langle S_r \rangle]_{\rm av} \sim r^{-(d-4+\overline{\eta})}$) correlation functions at T_c define two exponents η and $\overline{\eta}$. These are expected [291, 88] to be related to the new exponent θ via

$$\theta = 2 - \overline{\eta} + \eta \ . \tag{23}$$

And, as usual, one has the scaling relations $\gamma = \nu(2-\eta)$, $\beta = (d-4+\overline{\eta})$ and $\alpha + 2\beta + \gamma = 2$. Obviously there seem to be three independent critical exponents and a central issue of the activities on the critical properties of the RFIM is the quest for an additional scaling relation. Already Imry and Ma [133] conjectured that $F_{\rm sing} \sim \chi$, where $\chi \sim (T-T_c)^{-\gamma}$ is the susceptibility. This would imply $\theta = 2-\eta$ (21) and therefore via (23)

$$\overline{\eta} = 2\eta\ . \tag{24}$$

A set of analytical arguments by Schwartz et al. [252, 253, 254, 255] supports this two-exponent scaling scenario indicated by (24). Hence the exact Schwartz-Soffer inequality [253] $\overline{\eta} \leq 2\eta$ might be fulfilled as an equality. Results of numerical studies are compatible with this picture: In Monte Carlo simulations of the the diluted antiferromagnet in a uniform field (DAFF) Ogielski and Huse [210] found $\eta = 0.5 \pm 0.1$ and $\overline{\eta} = 1.0 \pm 0.3$. Ogielski [211] investigated ground state properties of the RFIM via combinatorial optimization methods [9] and found $\overline{\eta} = 1.1 \pm 0.1$ and $\nu = 1.0 \pm 0.1$. Rieger and Young [232] performed the most extensive Monte Carlo simulation of the RFIM up to now by sampling 1280 disorder configurations for each lattice size and temperature and obtained via finite size scaling (for $h/T = 0.35$)

$$\begin{array}{rcll} \eta & = & 0.56 & \pm 0.03 \\ \overline{\eta} & = & 1.00 & \pm 0.06 \\ \nu & = & 1.6 & \pm 0.3 \\ \gamma & = & 2.3 & \pm 0.3 \\ \beta & = & 0.00 & \pm 0.05 \\ \alpha & = & -1.0 & \pm 0.3 \end{array} \tag{25}$$

These are values for the binary $h_i = \pm h$ distribution of the random fields, however, those obtained for a continuous (Gaussian) distribution are not significantly different [233]. In addition Dayan et al. [77] performed a real space renormalization group calculation that gave identical values for η and $\overline{\eta}$ as those listed in (25). Finally an extensive high-temperature series expansion by Gofman et al. [105] find in three dimensions $\gamma = 2.1 \pm 0.2$, concurring with the Monte Carlo data (25) and $\overline{\gamma} = \nu(4-\overline{\eta}) = 2\gamma$, the last equality being a consequence of (24), which they showed to hold also in 4 and 5 dimensions. Moreover, they demonstrated an even stronger relation for the amplitude ratio to hold:

$$A = \lim_{T \to T_c} \frac{\chi_{\rm dis}}{\chi^2 (h/T)^2} = 1\ , \tag{26}$$

where $\chi_{\rm dis} = L^d [\langle S_i \rangle^2]_{\rm av}$ is the disconnected susceptibility. The Monte Carlo data can be analyzed in a similar way [233] and agree with (26). Thus there are strong indications for equation (24) to be correct and hence for the conjecture of two exponent scaling to be true.

A closer inspection of the list of exponents (25) shows some peculiarities: the order parameter exponent β turns out to be zero, which means that the magnetization jumps discontinuously to a non vanishing value at the critical temperature. This hints at a first order phase transition, a possibility that has already been suggested by Young and Nauenberg [301] (see however the interpretation of X-ray scattering studies by Hill et al. [122]). Note that $\beta = 0$ is also found for a continuous distribution [233], which is important since in case of a binary distribution for larger field strength a tricritical point is predicted in mean field theory [1], which separates a first order (high fields) from a second order (low fields) transition line in the $h - T$-diagram. Usually, at a first order phase transition, one would expect phase coexistence at T_c, which manifests itself in a typical multi peak structure in the probability distribution $P(m)$ for the order parameter [30, 56]. In the Monte Carlo data no indication for such a scenario can be found [233], instead $P(|m|)$ shows a significant peak at a nonzero value for $|m|$ already *at* the transition $T = T_c$ with $P(0) \to 0$ in the thermodynamic limit.

Furthermore, at a first order phase transition the specific heat usually diverges with the volume of the system [193], but the exponent α is *negative*, which means that the specific heat does *not* diverge at T_c. By means of birefringence techniques Belanger et al. [12, 13, 216] concluded from their experiments on $Fe_{0.47}Zn_{0.53}F_2$ that $\alpha = 0$ (i.e. a logarithmic divergence of the specific heat). Only recently it was shown [142] that the same kind of experiments on $Fe_{0.85}Mg_{0.15}Br_2$ are better compatible with a cusp like singularity of the specific heat and $\alpha = -1$, concurring with the value reported by in [232] see (25). However, this value for α together with the other estimates in (25) would violate the modified hyperscaling relation (22). Moreover, Schwartz [256] derived an exact inequality

$$2 - \alpha \leq \nu(d - 2 + \eta) \ , \tag{27}$$

and accepting $\eta \approx 0.5$ (since this result has a much smaller errorbar than ν) it would imply $\nu \geq 2$, which is a rather large value. Indeed in a recent Migdal-Kadanoff study Cao and Machta [50] find such a large exponent $\nu = 2.25$, and they also report $\alpha = -1.37$ and $\beta = 0.02$, consistent with (25). However, a value for α that is negative and large in modulus causes serious difficulties with respect to the Rushbroke-equality $\alpha + 2\beta + \gamma = 2$. Hence the random field enigma [261] is still far from being resolved.

In analogy to the spin glass research activity (see previous sections) there has been considerable interest in the non-equilibrium dynamics also in random field systems. It has been observed that diluted antiferromagnets fall into a metastable domain state if cooled in an external field B below the critical temperature $T_c(B)$ (see the article of Kleemann for a review [152]). This domain state has a finite correlation length that does not seem to increase with time, in contrast to the continuous aging phenomena observed in spin glasses (see section 3). Villain [290] and Grinstein and Fernandez [104] have predicted a logarithmic growth of the domain size (similar to what Fisher and Huse later predicted for spin glasses [91]), and numerical investigations of the domain growth in random field systems (e.g. [223, 221, 212]) seem to be compatible with this prediction (see also the review of Chowdhury and Biswal [62]). However, Nattermann and Vilfan [191] pointed out that DAFF map onto a RFIM *plus* random bonds, and the latter produce an enormous pinning force so that domain growth will only be observed on time scale up to 10^{11} years, which explains the experimental situation. Moreover, these time-persistent domains are fractal objects and have been analyzed in Monte Carlo simulations of DAFF by Nowak and Usadel [202, 203]. Recently also hysteresis effects in the RFIM at zero temperature have been studied [258, 76].

Nattermann and Vilfan also predict that after switching off the external field in a DAFF the magnetization concentrated in the domain walls will decay according to $M(t) \sim (\log t)^{-1/\Phi}$, which is compatible with experiments [161, 162]. However the results of Monte Carlo simulations fit better to an algebraic decay [200, 201] or an enhanced power law $M(t) \sim M_0 \exp\{-A(\log t/t')^y\}$ [204, 205].

Very frequently it can be observed that by field decreasing experiments with a DAFF a "stable" domain state already occurs for field strengths B larger than the value B_c below which the system orders antiferromagnetically (see [152] for an overview). Thus in between the paramagnetic and the ordered phase a region for this domain state has to be inserted in the corresponding phase diagram (see figure 6). This state has been identified with an intermediate spin glass phase, characterized by dynamical freezing and lack of long range antiferromagnetic order [189, 15]. From an experimental point of view (as well as for Monte Carlo simulations [202]) it is still not clear, whether this intermediate regime corresponds to an equilibrium phase. However, only recently Mézard and Young [182] looked at the N-component version of the RFIM in three dimensions and found in the limit $N \to \infty$ that replica symmetry is already broken *at* the ferromagnetic transition at temperature T_f (which is usually an indication for a spin glass phase). In a subsequent investigation Mézard and Monasson [184] were able to show that above T_f a glassy phase appears in a temperature interval $T_f < T < T_b$, T_b being the temperature above which the usual paramagnetic phase is entered. Interestingly the ferromagnetic correlation length turns out to be *finite*, reminiscent of a result by Guagnelli et al. [113], who study numerically the solutions of the mean field equations of the RFIM in three dimensions, and also very similar to the experimental finding of finite domain sizes in the intermediate domain state or spin glass region discussed above. Thus the phase diagram of RFIM (or DAFF) seems to be much richer than expected and in particular random field systems have much more in common with spin glasses than anticipated up to now.

6 Quantum spin glasses

The models discussed so far are all describing classical systems for which quantum fluctuations can be neglected. In most cases this is correct, namely as long as $T_c > 0$ since critical fluctuations at the transition occur at a frequency $\hbar\omega \ll k_B T$ that is proportional to the inverse relaxation time τ and thus approaches zero for $T \to T_c$ due to critical slowing down. Hence any finite temperature will destroy quantum coherence and the system will behave calassically. Very recently however, spin glasses began to enter the quantum regime [244].

The interesting theoretical question is: What are the effects of quantum mechanics on the physics of strongly disordered systems at zero temperature, where no heat bath is present and hopping over energy barriers is replaced by tunneling them quantum-mechanically. The best known and most studied example in this respect is the zero temperature metal-insulator transition [165]. The renewed interest in spin glasses in the quantum regime was kindled by a series of recent experiments [297, 298] on the dipolar Ising magnet $Li_xHo_{1-x}YF_4$, where T_c was driven down to zero by the application of a transverse magnetic field Γ (see the phase diagram depicted in figure 7). Experimental realizations of quantum spin glasses are already known for more than 10 years: the so called proton glasses [67, 215], which are random mixtures of ferroelectric and antiferroelectric materials like $Rb_{1-x}(NH_4)_xH_2PO_4$. Here the proton position, describable by an Ising spin variable, tunnels between two energy minima with a fixed frequency modelled by a transverse field acting on the spins. In the transverse field Ising magnets mentioned above, however, it became possible to study the *zero* temperature phase transition occuring for critical transverse field strength Γ_c by simply tuning the external field strength. This quantum phase transition lies within a different universality class than the usually studied (classical) spin glass transition at finite temperatures and it turns out that their properties differ significantly.

The above mentioned experiments can be described by the model Hamiltonian of an Ising spin glass in a transverse field [297, 298]

$$H = -\sum_{\langle ij \rangle} J_{ij}\sigma_i^z\sigma_j^z - \Gamma\sum_i \sigma_i^x \left(-h\sum_i \sigma_i^z\right) , \tag{28}$$

where the σ_i are Pauli spin matrices, Γ is the strength of the transverse field and h is a longitudinal magnetic field used to define magnetic susceptibilities but usually set to zero. Otherwise this model is equivalent to the EA-model in d dimensions defined in (1). Obviously, for $\Gamma = 0$ the quantum-mechanical Hamiltonian (28) is diagonal in the z-representaion of the spin operators, which in this case can simply be replaced by their eigenvalues ± 1 (after rescaling the couplings) giving exactly the classical EA-model (1). In this way the transverse field introduces the quantum mechanics into the spin glass problem and the value of Γ tunes the strength of the quantum fluctuations. At zero temperature and $\Gamma = 0$ the system described by (28) will be in its uniquely determined ground state, which is identical to the classical ground state of (1). In this case one has $\langle\sigma_i^z\rangle = \pm 1$ for all sites i and therefore $q_{EA} = [\langle\sigma_i^z\rangle^2]_{\rm av} = 1$, where $\langle\cdots\rangle$ means the quantum-mechanical expectation value.

If we switch on the transverse field ($\Gamma > 0$) the Hamiltonian (28) is not diagonal in the z-representation any more and its ground state will be a superposition of the classical ground state plus various excited states, which describes the quantum-mechanical tunneling at zero temperature between the local energy minima of the classical Hamiltonian. Furthermore $|\langle\sigma_i^z\rangle| < 1$ since the transverse field tries to align the spins in the x-direction and therefore $q_{EA} = [\langle\sigma_i^z\rangle^2]_{\rm av} < 1$. Increasing Γ diminishes the EA-order parameter q_{EA} and for some critical value Γ_c it will be zero: $q_{EA} = 0$ for $\Gamma \geq \Gamma_c$. This is the zero temperature phase transition we are interested in and obviously we cannot expect that its critical properties have anything in common with the finite temperature classical spin glass transition discussed in section 2.

In order to describe this zero-temperature transition one introduces a quantity measuring the distance from the critical transverse field strength (at $T = 0$) $\delta = (\Gamma - \Gamma_c)/\Gamma_c$. If one assumes a conventional second order phase transition one has $q_{EA} \sim |\delta|^\beta$ and $\chi_{SG} \sim |\delta|^{-\gamma}$ for $\delta \to 0$. Spatial correlations decay on a characteristic length scale that diverges at the critical point as usual: $\xi \sim |\delta|^{-\nu}$ and these exponents defined so far would be sufficient to describe the static critical behavior of a classical spin glass transition. However, at a zero temperature transition driven solely by quantum fluctuations static and dynamic quantities are linked in such way that the introduction of a characteristic time-scale (or inverse frequency) is necessary (see e.g. [94]):

$$\xi_\tau \sim \xi^z \sim |\delta|^{-z\nu} , \qquad (29)$$

where z is the dynamical exponent. This will become more evident below, when we consider an equivalent classical model.

Much work in the past has been devoted to the infinite range model [44, 135, 158, 287, 49, 106, 289] and the phase-diagram in the Γ-T-plane, which looks similar to the one shown in figure 7: for low enough temperature T and field Γ one finds a transition line separating a paramagnetic phase from a spin glass phase. Recently Miller and Huse [188] and Ye, Sachdev and Read [299] focused on the zero-temperature critical behavior and calculated the critical exponents $\gamma = 1/2$ (with multiplicative logarithmic corrections), $\beta = 1$ and $z\nu = 1/2$. Interestingly it seems that this quantum SK-model seems to fall into the same universality class as an infinite range metallic spin glass model incorporating itinerant electrons, which was investigated recently by Oppermann [214]. Another proposition for a mean-field quantum spin glass that is exactly solvable was made by Nieuwenhuizen [196] via the introduction of a quantum description of spherical spins.

In one dimension the situation is quite different. The transverse Ising chain with random bonds and/or random transverse fields can be mapped (see below) onto the McCoy-Wu model [176, 177, 178], for which various analytical results are known [123, 260, 194, 92, 186, 187]. In particular D. Fisher [92] has shown within a renormalization group calculation that *typical* and *average* spatial correlations behave differently: the typical correlation length diverges with an exponent $\tilde{\nu} = 1$ (as also found by Shankar and Murthy [260]), whereas the average correlation length diverges with an exponent $\nu = 2$, thus obeying the rigorous inequality $2/\nu \leq d = 1$ [59] as an equality. Due to activated dynamics it turns out that $\xi_\tau \sim \exp(A\sqrt{\xi})$ and therfore $z = \infty$. Furthermore it is predicted that $\beta = (3 - \sqrt{5})/2 \approx 0.38$ and that, similar to the surface magnetization [178], the bulk magnetization behaves nonanalytically as a function of the field h already above the critical point due to Griffith singularities, giving rise to a divergence of the linear and nonlinear susceptibility already in the disordered phase. In a recent finite size scaling analysis of results obtained via Monte Carlo simulations and the use of the transfer-matrix formalism [69] discrepancies to this scenario were found. This may well be a consequence of the smallness of the lattice sizes studied ($L \leq 16$), since for this situation one might expect *typical* rather than *average* results.

In two and three dimensions, which are most relevant for the above mentioned experiments, no analytical results are known — apart from renormalization group calculations using the Migdal-Kadanoff approximation [37, 66]. For this reason extensive Monte Carlo simulations have been performed recently in two dimensions by Rieger and Young [236] and in three dimensions by Guo, Bhatt and Huse [115]. Usually the investigation of quantum systems via Monte Carlo methods are hampered by various deficiencies, the sign problem being the most notorious one in this respect (see [168] for a review). In studying the Ising spin glass in a transverse field however, one can exploit the fact that it can be mapped exactly onto a classical Ising model described by a *real* Hamiltonian. Using the Suzuki-Trotter formula [282] one can easily show that the ground state energy of the d–dimensional quantum mechanical model (28) is equal to the free energy of a $(d+1)$–dimensional classical model, where the extra dimension corresponds to imaginary time, i.e.

$$\begin{aligned} -\frac{E(T=0)}{L^d} &= \lim\nolimits_{T\to 0} \frac{T}{L^d} \mathrm{Tr}\; e^{-\beta H} \\ &= \frac{1}{\Delta\tau}\frac{1}{L_\tau L^d} \mathrm{Tr}\; e^{-\mathcal{S}} \end{aligned} \tag{30}$$

where the imaginary time direction has been divided into L_τ time slices of width $\Delta\tau$ ($\Delta\tau L_\tau = \beta$), and the effective classical action, $\mathcal{S}$, is given by

$$\mathcal{S} = -\sum_\tau \sum_{\langle ij\rangle} K_{ij} S_i(\tau) S_j(\tau) - \sum_\tau \sum_i K S_i(\tau) S_i(\tau+1) \left(-\sum_\tau \sum_i H S_i(\tau) \right) , \tag{31}$$

where the $S_i(\tau) = \pm 1$ are classical Ising spins, the indices i and j run over the sites of the original d–dimensional lattice and $\tau = 1, 2, \ldots, L_\tau$ denotes a time slice. In equation (31),

$$\begin{aligned} K_{ij} &= \Delta\tau J_{ij} \\ H &= \Delta\tau h \\ \exp(-2K) &= \tanh(\Delta\tau\Gamma) \end{aligned} \tag{32}$$

One has the *same* random interactions in each time slice. In order to fulfill the second equality in (30) precisely, one has to perform the limit $\Delta\tau \to 0$, which implies $K_{ij} \to 0$ and $K \to \infty$. However, the universal properties of the phase transition are expected to be independent of $\Delta\tau$ so we take $\Delta\tau = 1$ and set the standard deviation of the K_{ij} to equal K. Thus K, which physically sets the relative strength of the transverse field and exchange terms in (28), is like an inverse "temperature" for the effective classical model in (31).

One sees that the (d+1)-dimensional classical model (31) should order at low "temperature" (or coupling constant K) like a spin glass in the d spatial dimensions and ferromagnetically in the imaginary time direction. From this one concludes the existence of two different diverging length-scales in the classical model (31): one for the spatial (spin glass)-correlations, which is ξ, and one for imaginary time (ferromagnetic) correlations, which is ξ_τ. Thus in the representation (31) the link between statics and dynamics in the original quantum model (28) becomes most obvious. Correspondingly, to analyze the critical properties of the extremely anisotropic classical model (31) one has to take into account these two length scales via anisotropic finite size scaling [32].

Monte Carlo simulations of the classical model (31) are straightforward — it turns out that sample-to-sample fluctuations are significant, for which reason one has to do an extensive disorder average [236, 115]. However, the finite size scaling analysis is complicated by the fact that due to the existence of two diverging length scales ξ and ξ_τ one has to deal with two independent scaling variables: as usual L/ξ and in addition the shape (or aspect ratio) L_τ/L^z of the system [32]. Thus, with the usual definition of a spin glass overlap $Q = L^{-d}L_\tau^{-1}\sum_{i,\tau} S_i^a(\tau)S_i^b(\tau)$ for the classical system, the dimensionless combination of moments of the order-parameter $g_{\rm av}$ obeys

$$g_{\rm av}(K, L, L_\tau) = 0.5[3 - \langle Q^4\rangle/\langle Q^2\rangle^2]_{\rm av} \sim \tilde{g}_{\rm av}(\delta L^{1/\nu}, L_\tau/L^z) \ . \tag{33}$$

In isotropic systems one has $z = 1$, which makes the aspect ratio constant to one for the choice $L = L_\tau$ and in order to determine the critical coupling K_c one exploits the fact that $g_{\rm av}(K, L, L)$ becomes independent of L for $K = K_c$ (see e. g. [23]). In the present case of a very anisotropic system z is not known a priori and one has to vary three different system parameters to obtain an estimate for K_c and z (and other exponents). The following method [236, 115] enhances the efficiency of such a search in a three-parameter space and also produces reliable estimates for the quantities of interest: In the limit $L_\tau \gg L^z$ the classical $(d+1)$-dimensional classical system is quasi-one-dimensional, and in the limit $L_\tau \ll L^z$ the system is quasi-d-dimensional and well above its transition "temperature" in d dimensions (which is even zero for $d = 2$). Therfore one has $\tilde{g}_{\rm av}(x, y) \to 0$ for $y \to 0$ and for $y \to \infty$. Hence, for fixed x, $\tilde{g}_{\rm av}(x, y)$ must have a maximum for some value $y = y_{\max}(x)$. The value of this maximum decreases with increasing L in the disordered phase $K < K_c$ (where $\delta = (K_c/K - 1) > 0$) and increases with increasing L in the ordered phase. This criterion can be used to estimate the critical coupling, as exemplified in figure (8). If one plots $g_{\rm av}(K_c, L, L_\tau)$ versus L_τ/L^z with the correct choice for the dynamical exponent z, one should obtain a data-collaps for all system sizes L. Finally one uses systems with fixed aspect ratio L_τ/L^z to determine critical exponents via the usual one–parameter finite size scaling.

Various scaling predictions can be made if one supposes a conventional second order phase transition to occur at some critical "temperature" K_c for the classical model. Hyperscaling in this particular situation would imply [94]

$$2 - \alpha = \nu(d + z) \ . \tag{34}$$

As usual one has $\gamma = \nu(2 - \eta)$, where η is defined via the decay of correlations at criticality:

$$\begin{aligned} C(r) &= [\langle S_i(\tau)S_{i+r}(\tau)\rangle^2]_{\rm av} &\sim& \quad r^{-(d+z-2+\eta)} \ , \\ G(t) &= [\langle S_i(\tau)S_i(t+\tau)\rangle]_{\rm av} &\sim& \quad r^{-(d+z-2+\eta)/2z} \ , \end{aligned} \tag{35}$$

and from (34) one gets via $\alpha + 2\beta + \gamma = 2$ the relation $2\beta/\nu = d + z - 2 + \eta$. The uniform magnetic susceptibility defined via

$$\chi_F = \frac{\partial[\langle\sigma_i^z\rangle]_{\rm av}}{\partial h} \sim |\delta|^{-\gamma_f} \tag{36}$$

with respect to the quantum mechanical Hamiltonian (28), is related to the integrated onsite correlation function of the classical model (31): $\chi_F \sim \sum_t G(t)$ and therefore $\gamma_f = \beta - \nu z$. Analogously the divergence of the magnetic nonlinear susceptibility for the quantum system

$$\chi_{nl} = \frac{\partial^3[\langle\sigma_i^z\rangle]_{\rm av}}{\partial h^3} \sim |\delta|^{-\gamma'} \tag{37}$$

can be estimated via the spin glass susceptibility of the classical model [236, 115] giving $\gamma' = \nu(2 - \eta + 2z)$.

In the following table we list the results obtained so far in various dimensions

	$d = 1$[92]	$d = 1$[69]	$d = 2$[236]	$d = 3$[115]	$d = 3$[37]	$d \geq 6$[299, 188]
z	∞	~ 1.7	1.50 ± 0.05	~ 1.3	~ 1.4	2
ν	2	~ 1	1.0 ± 0.1	~ 0.8	~ 0.87	1/4
η	0.38	~ 0.4	~ 0.5	~ 0.9	–	2
γ_f	*div.*	~ 2.3	~ 0.5	*finite*	–	*finite*
γ'	*div.*	–	~ 4.5	~ 3.5	–	0.5

(38)

The symbol *div.* in the second column means that first and higher derivatives of the magnetization diverge already in the disordered phase. The word *finite* means that in three dimensions and for d larger than the upper critical dimension the uniform susceptibility does not diverge at the critical point. The results in the second column [92] were obtained within a renormalization group calculation, those in columns 3 to 5 [69, 236, 115] with Monte Carlo simulations, column 6 shows the result of a Migdal-Kadanoff RNG calculation (in [66] also γ has been calculated in this way giving $\gamma = 1.2$, which is far off the MC-value) and the last column depicts analytical results from mean-field theory [299, 188].

Note that the results reported so far are for the zero-temperature critical behavior of the quantum model (28). To make contact with the recent experiments mentioned above, which are done in the vicinity of the quantum critical point at *finite* temperature, one can exploit the following scaling prediction. For finite temperatures the scaling variable of e.g. the nonlinear susceptibility is $\xi_\tau \cdot T$ since a finite temperature for the quantum system (28) implies a finite length $L_\tau \sim T^{-1}$ in the imaginary time direction of the classical model (31). Hence one has

$$\chi_{nl}(T,\delta) \sim \delta^{-\gamma'} \tilde{\chi}_{nl}\Big(\frac{T}{\delta^{z\nu}}\Big) \;, \tag{39}$$

with $\tilde{\chi}_{nl}(x) \to$ const. for $x \to 0$ and $\tilde{\chi}_{nl}(x) \to x^{-\gamma'/z\nu}$ for $x \to \infty$ (in order to cancel the divergent prefactor at finite T if $\delta \to 0$). Thus one has for $\Gamma = \Gamma_c$

$$\chi_{nl} \sim T^{-\gamma'/z\nu} \tag{40}$$

and analoguously e.g. $\chi_F \sim T^{-\gamma_F/z\nu}$. Equation (40) implies in three dimensions a strong divergence of the nonlinear susceptibility (when approaching $T = 0$) with a power ~ 2.7 that is very close to the one for classical Ising spin glasses (~ 2.9 see section 2). This is in striking contradiction to the observation made in the experiments mentioned in the introduction [298] and in order to clarify this discrepancy further work is necessary.

Another very important issue is the effect of Griffith singularities on the properties of quantum spin glasses. As already observed by McCoy and Wu [176, 178] for the one-dimensional quantum model (corresponding to the 2-dimensional classical model with layered disorder) their existence lead to divergencies in the magnetic susceptibility already in the disordered phase near the critical point (see also [260, 195, 92]). Thill and Huse [286] recently presented a droplet theory for quantum spin glasses, where they also predict algebraically decaying correlations in this region. This implies that the effect of Griffith singularities on the time-persistence of autocorrelations is much stronger for quantum systems at zero temperature than their effect for classical models at finite temperatures, where only an enhanced power law is predicted [222]. Preliminary results of Monte Carlo simulations of the two-dimensional transverse field Ising spin glass at zero temperature give already strong support for this scenario to be correct [239].

Finally we should mention that also the more general case of spin-$\frac{1}{2}$ Heisenberg model with quenched disorder, described by the Hamiltionian

$$H = \sum_{\langle ij \rangle} J_{ij} \{ \sigma_i^x \sigma_j^x + \delta_{ij} (\sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) \} - \sum_i \underline{h} \cdot \underline{\sigma} \qquad (41)$$

is of great interest, in particular because e.g. the copper-oxygen layers of high-T_c superconductors are supposed to be good physical realizations of such a model in two-dimensions [245]. The additional disorder-parameters δ_{ij} introduce a random anisotropy without destroying the XY-symmetry and the limit $\delta_{ij} \to 0$ reproduces the Ising case considered above. For vanishing external field $\underline{h} = 0$ the strength of the disorder $([J_{ij}^2]_{\rm av} - [J_{ij}]_{\rm av}^2)$ triggers the zero-temperature quantum phase transition of the model (41). For the isotropic case $\delta_{ij} = 1$ this scenario has been studied in two dimensions by Sandvik and Vekic [245] very recently with Monte Carlo simulations. Unfortunately, due to the notorious sign problem occuring in (Quantum)-Monte Carlo simulations, only the non-frustrated case ($J_{ij} > 0$) has been considered. On the other side, much progress could be made in the study of the XXZ-chain with quenched disorder (i.e. model (41) in one dimension [82, 116, 242]), which is well suited for exact diagonalization studies.

7 Other models and conclusion

Althought we have confined ourselves to *Ising* spin glasses in this review, we finally would like to mention some of the other spin glass like systems currently under investgation. A concise review on this matter has already been given by Young, Reger and Binder [302].

Many experimental spin glasses that most convincingly show features of a spin glass transition at finite temperatures are Heisenberg-like systems (e.g. CuMn and $Eu_xSr_{1-x}S$) described by the (classical) Hamiltonian

$$H = \sum_{\langle ij \rangle} J_{ij} \underline{S}_i \underline{S}_j \ , \qquad (42)$$

where $\underline{S}_i$ are 3-component vector spins. There seems to be ample numerical evidence (see [174] for references) that this model (with short range interactions) does *not* possess a finite T spin glass transition — which is an unsatisfactory discrepancy with the experimental situation (in four dimensions indications for a finite-T transition were found in a recent Monte Carlo study [65]). However, it has been demonstrated by Matsubara et al. [174] via numerical simulations using a hybrid Monte Carlo spin dynamics method that an anisotropy term modelled by

$$H_{\rm anis.} = \sum_{\langle ij \rangle} \sum_{\alpha \neq \beta} D_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta \qquad (43)$$

induces a finite temperature phase transition (see also the defect-wall renormalization-group study of Gingras [102]), which gives further support to the claim that all three-dimensional spin glasses with anisotropy should have a phase transition in the universality class of the short range Ising spin glass model [46].

Another generalization of the Ising models considered here are so called Potts glasses (see [302] for a review), where each spin can take on p values instead of only $p = 2$ in the Ising case. It seems that in three dimensions for $p = 3$ this model shows a spin glass transition only at zero temperature [246]. On the mean-field level it has been shown [149] that the p-state Potts glasses and spin glass models with p-spin interactions show very similar static and dynamic behavior for certain values of p. The question, whether the situation is similar in finite dimensions, has been discussed in [230], but due to the fact that on a finite dimensional lattice the definition of a model with p-spin interactions is to some extend arbitrary, it is not resolved yet. Furthermore it has been observed [149] that the dynamical saddle-point equations of both mean-field spin glass models (the Potts model and the p-spin model) have the same mathematical structure as that arising in mode-coupling theory for the structural glass transition [139]. This latter similarity also occurs for orientational glasses, (for a review see [34]), where the spins are replaced by quadrupolar moments.

Very similar to the XY spin glass is the so-called gauge glass defined by the Hamiltonian

$$H = \sum_{\langle ij \rangle} J_{ij} \cos(\varphi_i - \varphi_j - A_{ij}) \ , \tag{44}$$

where $\varphi \in [0, 2\pi[$ are phase-variables and $A_{ij} \in [0, 2\pi[$ is a quenched random vector potential. This model has been introduced by Shih et al. [263] to describe granular superconductors in an external field. It has been discussed that the gauge glass (44) might describe some of the low temperature physics of vortex-glasses (see [35], chapter 7 for a review), which is the proposed [93] "true" superconducting state of high-T_c materials. Of particular interest in this respect is the existence of a gauge glass transition in three dimensions. Monte Carlo simulations [130, 227, 228] and defect-wall energy analysis [101] of the gauge glass found indeed a phase transition in three dimensions at finite temperatures. Recently, however, it was demonstrated [38] that screening effects might destroy this transition. Moreover, it is not clear whether the gauge glass and the vortex glass really belong to the same universality class ([35], chapter 7). The latter has gained much interest for the analytically tractable case in 1+1 dimensions, which has been studied numerically via Monte Carlo simulations only very recently [11, 75]. Descrepancies to existing theories rekindled the interest in this model that still seems not to be fully understood.

Finally we would like to mention the numerical investigation of a Hubbard model of interacting bosons within a quenched random potential [94]. Similar to the case of the quantum spin glass discussed in the last section this quantum mechanical Hamiltonian can be mapped onto a real classical Hamiltonian in d+1 dimensions. Monte Carlo simulations of this model in two dimensions [275, 293] showed the existence of a transition from a super conducting phase to an insulating Bose glass phase.

In summary we have seen that the application and usefulness of Monte Carlo simulations of strongly disordered systems has spread from Ising spin glasses and random field systems to very diverse fields including structural glasses, superconductors and "dirty bosons". However, also the most "simple" spin glass model in finite dimensions, the EA-model with Ising spins, gains again increasing interest. Apart from many fundamental questions still being unsolved, like the existence of a transition within an external field, recent years' research activities began on one side to focus on quantum effects in spin glasses and on the other side to shift from critical behavior to non-equilibrium dynamics. In both cases the main stimulus came from experiments and Monte Carlo simulations play an important role in learning to understand the newly observed phenomena. Thus, to conclude, spin glasses are far from being dead — they are still aging.

Acknowledgement

During recent years I had the opportunity to discuss with many experts on this field. In this respect I would like to express my gratitude to J. Adler, D. Belanger, J. P. Bouchaud, D. Chowdhury, A. Crisanti, S. Franz, H. O. Heuer, H. Horner, D. Huse, W. Kinzel, W. Kleemann, L. Mikheev, T. Nieuwenhuizen, P. Nordblad, M. Schreckenberg, P. Sibani, D. Stauffer, T. Nattermann, E. Vincent and A. P. Young. My special thanks also to J. Kisker and B. Steckemetz. This work work was performed within the SFB 341 Köln–Aachen–Jülich, supported by the DFG.

References

[1] A. Aharony, Phys. Rev. B **18**, 3318 (1978).

[2] M. Alba, M. Ocio and J. Hammann, Europhys. Lett. **2**, 45 (1986).

[3] J. R. de Almeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).

[4] J. O. Andersson, J. Mattson and P. Svedlindh, Phys. Rev. B **46**, 8297 (1992).

[5] J. O. Andersson, J. Mattson and P. Svedlindh, Phys. Rev. B **49**, 1120 (1994).

[6] J. O. Andersson and P. Sibani, preprint (1994).

[7] L. de Arcangelis, A. Coniglio and H. J. Herrmann, Europhys. Lett. **9**, 749 (1989).

[8] D. Badoni, J. C. Ciria, F. Ritort, J. Pech and J. J. Ruiz-Lorenzo, Europhys. Lett. **21**, 495 (1993).

[9] F. Barahona, J. Phys. A **15**, 3241 (1982).

[10] F. Barahona, Phys. Rev. B **49** 12864 (1994).

[11] G. G. Batrouni and T. Hwa, Phys. Rev. Lett. **72**, 4133 (1994).

[12] D. P. Belanger, A. R. King, V. Jaccarino and J. L. Cardy, Phys. Rev. B **28**, 2522 (1983).

[13] D. P. Belanger, A. R. King and V. Jaccarino, Phys. Rev. B **31**, 4538 (1985).

[14] D. P. Belanger and A. P. Young, J. Magn. Magn. Mat. **100**, 272 (1991).

[15] D. P. Belanger, W. E. Murray, F. C. Montenegro, A. R. King, V. Jaccarino and R. W. Erwin, Phys. Rev. B **44**, 2161 (1991).

[16] L. Bernardi and I. A. Campbell, Phys. Rev. B **49**, 728 (1994).

[17] L. Bernardi and I. A. Campbell, Europhys. Lett. **26**, 147 (1994).

[18] B. A. Berg and T. Neuhaus, Phys. Rev. Lett. **68**, 9 (1992).

[19] B. A. Berg and T. Celik, Phys. Rev. Lett. **69**, 2292 (1992).

[20] B. A. Berg, T. Celik and U. Hansmann, Europhys. Lett. **22**, 63 (1993).

[21] B. A. Berg, U. Hanssmann and T. Celik, Int. J. Mod. Phys. C **5**, 263 (1994).

[22] R. N. Bhatt and A. P. Young, Phys. Rev. Lett. **54**, 924 (1985).

[23] R. N. Bhatt and A. P. Young in [119].

[24] R. N. Bhatt and A. P. Young, Phys. Rev. B **37**, 5606 (1988).

[25] R. N. Bhatt and A. P. Young, Europhys. Lett. **20**, 59 (1992).

[26] K. Biljakovic, J. C. Lasjaunias, P. Monceau and F. Levy, Phys. Rev. Lett. **62**, 1512 (1989).

[27] K. Biljakovic, J. C. Lasjaunias, P. Monceau and F. Levy, Phys. Rev. Lett. **67**, 1902 (1991).

[28] K. Binder and K. Schroeder, Phys. Rev. B **24**, 2142 (1976).

[29] K. Binder in *Monte Carlo Methods in Statistical Physics*, K. Binder ed., Topics in Current Physics **7**, Springer-Verlag, Berlin Heidelberg (1979)

[30] K. Binder and D. P. Landau, Phys. Rev. B **30**, 1477 (1984).

[31] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).

[32] K. Binder and J. S. Wang, J. Stat. Phys. **55**, 87 (1989).

[33] K. Binder (ed.): *The Monte Carlo Method in Condensed Matter Physics*, Topics in Applied Physics **71**, Springer-Verlag, Berlin-Heidelberg-New York (1992).

[34] K. Binder and J. D. Reger, Advances in Physics **41**, 547 (1992).

[35] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A. I. Larkin and V. M. Vinokur, Rev. Mod. Phys. in press (1994).

[36] R. E. Blundell, K. Humayun and A. J. Bray, J. Phys. A **25**, L733 (1992).

[37] B. Boechat, R. R. dos Santos and M. A. Continentino, Phys. Rev. B **49**, 6404 (1994).

[38] H. S. Bokil and A. P. Young, preprint cond-mat/9408029 (1994).

[39] A. B. Bortz, M. H. Kalos and J. L. Lebowitz, J. Comp. Phys. **17**, 10 (1975).

[40] J. P. Bouchaud, J. Physique I **2**, 1705 (1992).

[41] J. P. Bouchaud, E. Vincent and J. Hammann, J. Physique I **4**, 139 (1994).

[42] J. P. Bouchaud and M. Mézard, J. Physique I **4**, 1109 (1994).

[43] J. P. Bouchaud and D. S. Dean, preprint cond-mat/9410022 (1994).

[44] A. J. Bray and M. A. Moore, J. Phys. C **13**, L655 (1980).

[45] A. J. Bray and M. A. Moore, J. Phys. C **17**, L463 (1984).

[46] A. J. Bray, M. A. Moore and A. P. Young, Phys. Rev. Lett. **56**, 2641 (1986).

[47] A. J. Bray and M. A. Moore, Phys. Rev. Lett. **58**, 57 (1987).

[48] J. Bricmont and A. Kupiainen, Phys. Rev. Lett. **59** 1829 (1987).

[49] G. Büttner and K. D. Usadel, Phys. Rev. B **41**, 428 (1990).

[50] M. S. Cao and J. Machta, Phys. Rev. B **48**, 3177 (1993).

[51] S. Caracciolo, G. Parisi, S. Parternello and N. Sourlas, Europhys. Lett. **11**, 783 (1990).

[52] S. Caracciolo, G. Parisi, S. Parternello and N. Sourlas, J. Physique **51**, 1877 (1990).

[53] S. Caracciolo, G. Parisi, S. Parternello and N. Sourlas, J. Physique I **1**, 627 (1991).

[54] S. Caracciolo, Physica A **185**, 261 (1992).

[55] J. Cardy, Phys. Rev. B **29**, 505 (1985).

[56] M. S. S. Challa, D. P. Landau and K. Binder, Phys. Rev. B **34**, 1841 (1986).

[57] I. A. Campbell and L. de Arcangelis, Europhys. Lett. **13**, 587 (1990).

[58] P. Chandra, P. Coleman and I. Ritchey, J. Physique I **3**, 591 (1993).

[59] J. T. Chayes, L. Chayes, D. S. Fisher and T. Spencer, Phys. Rev. Lett. **57**, 299 (1986).

[60] H. F. Cheung and W. L. McMillan, J. Phys. C **16**, 7027 (1983); 7033 (1983).

[61] D. Chowdhury, *Spin glasses and other frustrated systems*, World Scientific, Singapore (1986).

[62] D. Chowdhury and B. Biswal, p. 55 in [278]

[63] J. C. Ciria, G. Parisi and F. Ritort, J. Phys. A **26**, 6731 (1993).

[64] J. C. Ciria, G. Parisi, F. Ritort and J. J. Ruiz-Lorenzo, J. Physique I **3**, 2207 (1993).

[65] B. Coluzzi, preprint cond-mat/9410053 (1994).

[66] M. A. Continentino, B. Boechat and R. R. dos Santos, preprint (1994).

[67] E. Courtens, Phys. Rev. Lett. **52**, 69 (1984).

[68] A. Crisanti, H. Horner and H. J. Sommers, Z. Phys. B **92**, 257 (1993).

[69] A. Crisanti and H. Rieger, J. Stat. Phys. **77** 1087 (1994).

[70] L. Cugliandolo and J. Kurchan, Phys. Rev. Lett. **71**, 173 (1993).

[71] L. Cugliandolo, J. Kurchan and F. Ritort, Phys. Rev. B **49**, 6331 (1994).

[72] L. Cugliandolo and J. Kurchan, J. Phys. A **27**, 5749 (1994).

[73] L. F. Cugliandolo, J. Kurchan and G. Parisi, preprint cond-mat/9406053 (1994).

[74] L. F. Cugliandolo, J. Kurchan, G. Parisi and F. Ritort, preprint cond-mat/9407086 (1994)

[75] D. Cule and Y. Shapir, preprint (1994).

[76] K. Dahmen and J. P. Sethna, Phys. Rev. Lett. **71**, 3222 (1993).

[77] I. Dayan, M. Schwartz and A. P. Young, J. Phys. A. **26**, 3093 (1993).

[78] C. Dekker, A. F. M. Arts, H. W. de Wijn, A. J. van Duyneveldt and J. A. Mydosh, Phys. Rev. Lett. **61**, 1780 (1988).

[79] C. Dekker, A. F. M. Arts, H. W. de Wijn, A. J. van Duyneveldt and J. A. Mydosh, Phys. Rev. B **40**, 11243 (1989).

[80] B. Derrida and G. Weisbuch, Europhys. Lett. **4**, 657 (1987).

[81] V. S. Dotsenko, W. Selke and A. L. Talapov, Physica A **170**, 278 (1991).

[82] C. A. Doty and D. S. Fisher, Phys. Rev. B **45**, 2167 (1992).

[83] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[84] H. Eissfeller and M. Opper, Phys. Rev. Lett. **68**, 2094 (1992).

[85] H. Eissfeller and M. Kinzel, J. Phys. A **25**, 1473 (1992).

[86] A. M. Ferrenberg and R. H. Swendsen, Phys. Rev. Lett. **63**, 1195 (1989).

[87] K. H. Fischer and J. A. Hertz, *Spin glasses*, Cambridge University Press, Cambridge (1991).

[88] D. S. Fisher, Phys. Rev. Lett. **56**, 416 (1986).

[89] D. S. Fisher and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986).

[90] D. S. Fisher and D. A. Huse, Phys. Rev. B **38**, 373 (1988).

[91] D. S. Fisher and D. A. Huse, Phys. Rev. B **38**, 386 (1988).

[92] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992).

[93] M. P. A. Fisher, Phys. Rev. Lett. **62**, 1415 (1989).

[94] M. P. A. Fisher, P. B. Weichman, G. Grinstein and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).

[95] S. Fishman and A. Aharony, J. Phys. C **12**, L729 (1979).

[96] S. Franz and M. Mézard, Physica A **210**, 48 (1994).

[97] S. Franz and H. Rieger, cond-mat/9410112, J. Stat. Phys. in press (1994).

[98] S. Franz and J. Hertz, preprint cond-mat/9408079.

[99] M. Freixa–Pascual and H. Horner, Z. Phys. B **80**, 95 (1990).

[100] T. R. Gawron, M. Cieplak and J. R. Banavar, J. Phys. A **24**, L127 (1991).

[101] M. J. P. Gingras, Phys. Rev. B **45**, 7547 (1992).

[102] M. J. P. Gingras, Phys. Rev. Lett. **71**, 1637 (1993).

[103] G. S. Grest and D. Srolovitz, Phys. Rev. B **32**, 3014 (1985).

[104] G. Grinstein and J. F. Fernandez, Phys. Rev. B **29**, 6389 (1984).

[105] M. Gofman, J. Adler, A. Aharony, A. B. Harris et al., Phys. Rev. Lett. **71**, 1569 (1993).

[106] Y. Y. Goldschmidt and P. Y. Lai, Phys. Rev. Lett. **64**, 2567 (1990).

[107] P. Granberg, P. Svedlindh, P. Nordblad, L. Lundgren and H. S. Chen, Phys. Rev. B **35**, 2075 (1987).

[108] P. Granberg, L. Sandlund, P. Nordblad, P. Svedlindh and L. Lundgren, Phys. Rev. B **38**, 7097 (1988).

[109] P. Granberg, L. Lundgren and P. Nordblad, J. Magnetism and Magnetic Materials **92**, 228 (1990).

[110] E. R. Grannan and R. E. Hetzel, Phys. Rev. Lett. **67**, 907 (1991).

[111] M. Grötschel, M. Jünger and G. Reinelt, in [119].

[112] U. Gropengiesser, J. Physique I **4** 1367 (1994).

[113] M. Guagnelli, E. Marinari and G. Parisi, J. Phys. A **26**, 5675 (1993).

[114] K. Gunnarson, P. Svedlindh, P. Nordblad, L. Lundgren, H. Aruga and A. Ito, Phys. Rev. Lett. **61**, 754 (1988).

[115] M. Guo, R. N. Bhat and D. Huse, Phys. Rev. Lett. **72**, 4137 (1994).

[116] S. Haas, J. Riera and E. Dagotto, Phys. Rev. B **48**, 13174 (1993).

[117] A. B. Harris, T. C. Lubensky and J. H. Chen, Phys. Rev. Lett. **36**, 415 (1976).

[118] J. L. van Hemmen and I. Morgenstern (eds.), *Heidelberg Colloquium on Spin Glasses*, Lecture Notes in Physics **192**, Springer, Berlin (1983).

[119] J. L. van Hemmen and I. Morgenstern (eds.), *Heidelberg Colloquium on Spin Glasses*, Lecture Notes in Physics **275**, Springer, Berlin (1986).

[120] R. E. Hetzel, R. N. Bhatt and R. R. P. Singh, Europhys. Lett. **22**, 383 (1993).

[121] H. O. Heuer, Comp. Phys. Comm. **59**, 387 (1990).

[122] J. P. Hill, Q. Feng, R. J. Birgeneau and T. R. Thurston, Z. Phys. B **92**, 285 (1993).

[123] P. Hoever, W. F. Wolff and J. Zittartz, Z. Phys. B **41**, 43 (1981).

[124] K. H. Hoffmann, T. Meintrup, P. Sibani and C. Uhlig, Europhys. Lett. **22**, 565 (1993).

[125] T. Hogg, this volume (1994).

[126] R. Hoogerbeets, Wei–Li Luo and R. Orbach, Phys. Rev. Lett. **55**, 111 (1985).

[127] K. Hukushima and K. Nemoto, J. Phys. C **5**, 1389 (1993).

[128] D. Huse and I. Morgenstern, Phys. Rev. B **32**, 3032 (1985).

[129] D. Huse, Phys. Rev. B **40**, 304 (1989).

[130] D. A. Huse and H. S. Seung, Phys. Rev. B **42**, 1059 (1990).

[131] D. Huse, Phys. Rev. B **43**, 8677 (1991).

[132] D. A. Huse and D. S. Fisher, J. Physique I **1**, 621 (1991).

[133] Y. Imry and S. K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).

[134] L. B. Ioffe, Phys. Rev. B **38**, 5181 (1988).

[135] H. Ishii and T. Yamamoto, J. Phys. C **18**, 6225 (1985). .

[136] A. Ito, H. Aruga, E. Torikai, M. Kikuchi, Y. Syono and H. Takai, Phys. Rev. Lett. **57**, 483 (1986).

[137] N. Ito and Y. Kanada, Supercomputer **25**, 31 (1988).

[138] N. Ito, Int. J. Mod. Phys. C **4**, 1131 (1994).

[139] J. Jäckle, Rep. Prog. Phys. **49**, 171 (1986);

[140] N. Jan and T. S. Ray, J. Stat. Phys. **75**, 1197 (1994).

[141] N. Jan and L. Arcangelis, p. 1 in [278].

[142] M. Karszewiski, J. Kushauer, Ch. Binek, W. Kleemann and D. Bertrand, J. Phys. C **6**, L75 (1994).

[143] N. Kawashima and M. Suzuki, J. Phy. A **25**, 1055 (1992).

[144] N. Kawashima, N. Ito and M. Suzuki, J. Phys. Soc. Jap. **61**, 1777 (1992).

[145] N. Kawashima and N. Ito, J. Phys. Soc. Jap. **62**, 435 (1993).

[146] W. Kinzel, Phys. Rev. B **19**, 4595 (1979).

[147] W. Kinzel, Phys. Rev. B **33**, 5086 (1986).

[148] H. Kinzelbach and H. Horner, J. Physique I **3**, 1329 & 1901 (1993).

[149] T. R. Kirkpatrick and D. Thirumalai, Phys. Rev. B **36**, 5388 (1987).

[150] J. Kisker, H. Rieger and M. Schreckenberg, cond-mat/9408060, J. Phys. A, in press (1994).

[151] J. Kisker, H. Rieger and M. Schreckenberg, to be published (1994).

[152] W. Kleemann, Int. J. Mod. Phys. B **7**, 2469 (1993).

[153] L. Klein, J. Adler, A. Aharony, A. B. Harris and Y. Meir, Phys. Rev. B **43**, 11249 (1991).

[154] T. Klotz and S. Kobe, J. Phys. A **27**, L95 (1994).

[155] T. Klotz and S. Kobe, Acta Phys. slov. **44**, 347 (1994).

[156] I. Kondor, J. Phys. A **22**, L163 (1989).

[157] G. J. M. Koper and H. J. Hilhorst, J. Phys. France **49**, 429 (1988).

[158] T. K. Kopec, K. D. Usadel and G. Büttner, Phys. Rev. B **39**, 12418 (1988).

[159] W. Krauth and O. Pulchery, J. Phys. A **27**, L715 (1994).

[160] W. Krauth and M. Mézard, cond-mat/9407029, Z. Phys. B, in press (1994).

[161] J. Kushauer, C. Binek and W. Kleemann, J. Appl. Phys. **75**, 5856 (1994).

[162] J. Kushauer, W. Kleemann, J. Mattson and P. Nordblad, Phys. Rev. B **49**, 6346 (1994).

[163] Z. W. Lai, G. F. Mazenko and O. T. Valls, Phys. Rev. B **37**, 9481 (1988).

[164] M. Ledermann, R. Orbach, J. M. Hammann, M. Ocio and E. Vincent, Phys. Rev. B **44**, 7403 (1991).

[165] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[166] F. Lefloch, J. Hammann, M. Ocio and E. Vincent, Europhys. Lett. **18**, 647 (1992).

[167] S. Liang, Phys. Rev. Lett. **69**, 2145 (1992).

[168] W. von der Linden, Phys. Rep. **222**, 53 (1992).

[169] L. Lundgren, R. Svedlindh, P. Nordblad and O. Beckman, Phys. Rev. Lett. **51**, 911 (1983).

[170] L. Lundgren, P. Nordblad, R. Svedlindh and O. Beckman, J. Appl. Phys. **57**, 3371 (1985).

[171] E. Marinari and G. Parisi, Europhys. Lett. **19**, 451 (1992).

[172] E. Marinari, G. Parisi and F. Ritort, J. Phys. A **27**, 2687 (1994).

[173] E. Marinari, G. Parisi and F. Ritort, preprint hep-th/9405148 and cond-mat/9406074.

[174] F. Matsubara, T. Iyota and S. Inawashiro, Phys. Rev. Lett. **67**, 1458 (1991).

[175] J. Mattson, C. Djurberg, P. Nordblad, L Hoines, R. Stubi and J. A. Cowen, Phys. Rev. B **47**, 14626 (1993).

[176] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968).

[177] B. M. McCoy and T. T. Wu, Phys. Rev. **188**, 982 (1969).

[178] B. M. McCoy and T. T. Wu, Phys. Rev. **188**, 1014 (1969).

[179] W. L. McMillan, Phys. Rev. B **28**, 5216 (1983).

[180] W. L. McMillan, Phys. Rev. B **29**, 4026 (1984). **30**, 476 (1984);

[181] M. Mézard, G. Parisi and M. A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific (1987).

[182] M. Mézard and A. P. Young, Europhys. Lett. **18**, 653 (1992).

[183] M. Mézard and S. Franz, Europhys. Lett. **26**, 209 (1994).

[184] M. Mézard and R. Monasson, Phys. Rev. B **50**, 7199 (1994).

[185] G. Migliorni and F. Ritort, preprint cond-mat/9407105 (1994).

[186] L. V. Mikheev and M. E. Fisher, Phys. Rev. B **49**, 378 (1994).

[187] L. V. Mikheev, Phys. Rev. B, in press (1994).

[188] J. Miller and D. Huse, Phys. Rev. Lett. **70**, 3147 (1993).

[189] F. C. Montenegro, A. R. King, V. Jaccarino, S. J. Han and D. P. Belanger, Phys. Rev. B **44**, 2155 (1991).

[190] J. A. Mydosh: *Spin glasses: an experimental introduction*, Taylor & Francis, London (1993).

[191] T. Nattermann and I. Vilfan, Phys. Rev. Lett. **61**, 223 (1988).

[192] T. Nattermann and J. Villain, Phase Transitions **11**, 817 (1988).

[193] M. Nauenberg and B. Nienhuis, Phys. Rev. Lett. **33**, 944 (1974).

[194] T. M. Nieuwenhuizen and H. Orland, Phys. Rev. B **40**, 5094 (1989).

[195] T. M. Nieuwenhuizen and H. Orland, Phys. Rev. Lett. **63**, 1760 (1989).

[196] T. M. Nieuwenhuizen, preprint cond-mat/9408055 and cond-mat/9408056 (1994).

[197] M. Ney-Nifle and H. J. Hilhorst, Phys. Rev. Lett. **68**, 2992 (1992).

[198] M. Ney-Nifle and H. J. Hilhorst, Physica A **193**, 48 (1993).

[199] P. Nordblad, P. Svedlindh, L. Lundgren and L. Sandlund, Phys. Rev. B **33**, 645 (1986).

[200] U. Nowak and K. D. Usadel, Phys. Rev. B **39**, 2516 (1989).

[201] U. Nowak and K. D. Usadel, Phys. Rev. B **43**, 851 (1991).

[202] U. Nowak and K. D. Usadel, Phys. Rev. B **44**, 7426 (1991).

[203] U. Nowak and K. D. Usadel, Phys. Rev. B **46**, 8329 (1992).

[204] S. J. Han, D. Belanger, W. Kleemann and U. Nowak, Phys. Rev. B **45**, 9728 (1992).

[205] S. J. Han and D. Belanger, Phys. Rev. B **46**, 2926 (1992).

[206] M. Ocio, M. Alba and J. Hamman, J. Physique Lett. **46**, L1101 (1985).

[207] A. T. Ogielski and D. L. Stein, Phys. Rev. Lett. **55**, 1634 (1985).

[208] A. T. Ogielski and I. Morgenstern, Phys. Rev. Lett. **54**, 928 (1985).

[209] A. T. Ogielski, Phys. Rev. B **32**, 7384 (1985).

[210] A. T. Ogielski and D. A. Huse, Phys. Rev. Lett. **56**, 1298 (1986).

[211] A. T. Ogielski, Phys. Rev. Lett. **57**, 1251 (1986).

[212] E. Oguz, J. Phys. A **27**, 2985 (1994).

[213] P. M. C. de Olivera: *Computing Boolean Statistical Models*, World Scientific Publ., Singapore (1991).

[214] R. Oppermann and M. Binderberger, Ann. Phys. **3**, 494 (1994).

[215] R. Pirc, B. Tadić and R. Blinc, Phys. Rev. B **36**, 8607 (1987).

[216] P. Pollack, W. Kleemann and D. P. Belanger, Phys. Rev. B **38**, 4773 (1988).

[217] I. Morgenstern and K. Binder, Phys. Rev. Lett. **57**, 1615 (1979).

[218] G. Parisi and F. Ritort, J. Physique I **3**, 969 (1993).

[219] G. Parisi and E. Marinari, J. Phys. A **26**, L1149 (1993).

[220] G. Parisi and F. Ritort, J. Phys. A **26**, 6711 (1993).

[221] S. Puri and N. Parekh, J. Phys. A **26**, 2777 (1993).

[222] M. Randeira, J. P. Sethna and R. G. Palmer, Phys. Rev. Lett. **54**, 1321 (1985).

[223] M. Rao and A. Chakrabarti, Phys. Rev. Lett. **71**, 3501 (1993).

[224] T. S. Ray and N. Jan, J. Physique I **4**, 2125 (1993).

[225] P. Refregier, E. Vincent, J. Hammann and M. Ocio, J. Physique **46**, 1533 (1987).

[226] J. D. Reger, R. N. Bhatt and A. P. Young, Phys. Rev. Lett. **64**, 1859 (1990).

[227] J. D. Reger, T. A. Tokuyasu, A. P. Young and M. P. A. Fisher, Phys. Rev. B **44**, 7147 (1991).

[228] J. D. Reger and A. P. Young, J. Phys. A **26**, L1067 (1993).

[229] J. N. Reimers and A. J. Berlinsky, Phys. Rev. B **48**, 9539 (1993).

[230] H. Rieger, Physica A **184**, 279 (1992).

[231] H. Rieger, J. Stat. Phys. **70**, 1063 (1993).

[232] H. Rieger and A. P. Young, J. Phys. A **26**, 5279 (1993).

[233] H. Rieger, unpublished results.

[234] H. Rieger, J. Phys. A **26**, L615 (1993).

[235] H. Rieger, J. Physique I **4**, 883 (1994).

[236] H. Rieger and A. P. Young, Phys. Rev. Lett. **72**, 4141 (1994).

[237] H. Rieger, B. Steckemetz and M. Schreckenberg, Europhys. Lett. **27**, 485 (1994).

[238] H. Rieger, J. Kisker and M. Schreckenberg, Physica A **210**, 324 (1994).

[239] H. Rieger and A. P. Young (in preparation).

[240] F. Ritort, Phys. Rev. B **50**, 6844 (1994).

[241] C. Rossel, Y. Maeno and I. Morgenstern, Phys. Rev. Lett. **62**, 681 (1989).

[242] K. J. Runge and G. T. Zimanyi, Phys. Rev. B **49**, 15212 (1994).

[243] D. H. Ryan (Ed.) *Recent Progress in Random Magnets* (World Scientific, Singapore, 1992).

[244] S. Sachdev, Physics World **7**, 25 (1994).

[245] A. W. Sandvik and M. Vekić, preprint cond-mat9406025 (1994).

[246] M. Scheucher, J. D. Reger, K. Binder and A. P. Young, Phys. Rev. B **42**, 6881 (1990).

[247] A. Schins, A. F. M. Arts and H. W. de Wijn, Phys. Rev. Lett. **70**, 2340 (1993).

[248] A. Schins, E. M. Dons, A. F. M. Arts, H. W. de Wijn, E. Vincent, L. Leylekian and J. Hammann, Phys. Rev. B **48**, 16524 (1993).

[249] M. Schreckenberg, Z. Phys. B **60**, 483 (1985).

[250] M. Schreckenberg and H. Rieger, in preparation (1994).

[251] C. Schulze, K. H. Hoffmann and P. Sibani, Europhys. Lett. **15**, 361 (1991).

[252] M. Schwartz, J. Phys. C **18**, 135 (1985).

[253] M. Schwartz and A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985).

[254] M. Schwartz and A. Soffer, Phys. Rev. B **33**, 2059 (1986).

[255] M. Schwartz, M. Gofman and T. Nattermann, Physica A **178**, 6 (1991).

[256] M. Schwartz, Europhys. Lett. **15**, 777 (1994).

[257] W. Selke, L. N. Schur and A. L. Talapov, p. 17 in [278].

[258] J. P. Sethna, K. Dahmen, S. Kartha, J. A. Krumhansl, B. W. Roberts and J. D. Shore, Phys. Rev. Lett. **70**, 3347 (1993).

[259] P. J. Shah and O. G. Mouritsen, Phys. Rev. B **41**, 7003 (1990).

[260] R. Shankar and G. Murthy, Phys. Rev. B **36**, 536 (1987).

[261] Y. Shapir in [243].

[262] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1972 (1975).

[263] W. Y. Shih, C. Ebner and D. Stroud, Phys. Rev. B **30**, 134 (1984).

[264] J. D. Shore, M. Holzer and J. P. Sethna, Phys. Rev. B **46**, 11376 (1992).

[265] P. Sibani and K. H. Hoffmann, Phys. Rev. Lett. **63**, 2853 (1989).

[266] P. Sibani, J. C. Schön, P. Salamon and J. A. Andersson, Europhys. Lett. **22**, 479 (1993).

[267] P. Sibani and P. Schriver, Phys. Rev. B **49**, 6667 (1994).

[268] P. Sibani and J. O. Andersson, Physica A **206**, 1 (1994).

[269] C. de Simone, G. Rinaldi, M. Diehl, M. Jünger, P. Mutzel and G. Reinelt, J. Stat. Phys. in press (1994)

[270] R. R. P. Singh and S. Chakravarty, Phys. Rev. Lett. **57**, 245 (1986).

[271] R. R. P. Singh and S. Chakravarty, Phys. Rev. B **36**, 547 and 559 (1987).

[272] R. R. P. Singh and D. Huse, J. Appl. Phys. 69, 5225 (1991).

[273] H. J. Sommers, Phys. Rev. Lett. **58**, 1268 (1987).

[274] H. Sompolinsky und A. Zippelius, Phys. Rev. Lett. **47**, 359 (1981).

[275] E. S. Sorensen, M. Wallin, S. M. Girvin and A. P. Young, Phys. Rev. Lett. **69**, 828 (1994).

[276] D. A. Stariolo and C. Tsallis, this volume (1994).

[277] D. Stauffer, J. Phys. A **26**, L525 (1993).

[278] D. Stauffer (ed.:) *Annual Reviews of Comp. Phys. I*, World Scientific, Singapore (1994).

[279] M. Stevens, M. Cleary and D. Stauffer, Physica A **208**, 1 (1994).

[280] L. C. E. Struik, *Physical Aging in Amorphous Polymers and other materials*, Elsevier, North-Holland (1978).

[281] P. Sutton, D. L. Hunter, N. Jan, J Physique I **4**, 1281 (1994).

[282] M. Suzuki, Progr. Theor. Phys. **56**, 1454 (1976).

[283] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. **57**, 2606 (1986).

[284] R. H. Swendsen and J. S. Wang, Phys. Rev. Lett. **58**, 86 (1987).

[285] R. H. Swendsen, J. S. Wang and A. M. Ferrenberg, in [33]

[286] M. J. Thill and D. A. Huse, preprint (1994).

[287] D. Thirumalai, Q. Li and T. R. Kirkpatrick, J. Phys. A **22**, 3339 (1989).

[288] M. Tomassini, this series (1995).

[289] K. D. Usadel, G. Büttner and T. Kopec, Phys. Rev. B **44**, 12583 (1991).

[290] J. Villain, Phys. Rev. Lett. **52**, 2606 (1984).

[291] J. Villain, J. Physique **46**, 1843 (1985).

[292] E. Vincent, J. Hammann and M. Ocio, in [243].

[293] M. Wallin, E. S. Sorensen, S. M. Girvin and A. P. Young, Phys. Rev. B **49**, 12115 (1994).

[294] J. S. Wang and R. H. Swendsen, Phys. Rev. **38**, 9086 (1988).

[295] J. Wang and A. P. Young, J. Phys. A **26**, 1063 (1993).

[296] M. B. Weissmann, Rev. Mod. Phys. **65**, 829 (1993).

[297] W. Wu, B. Ellmann, T. F. Rosenbaum, G. Aeppli and D. H. Reich, Phys. Rev. Lett. **67**, 2076 (1991).

[298] W. Wu, D. Bitko, T. F. Rosenbaum and G. Aeppli, Phys. Rev. Lett. **71**, 1919 (1993).

[299] J. Ye, S. Sachdev and N. Read, Phys. Rev. Lett. **70**, 4011 (1993).

[300] A. P. Young, Phys. Rev. Lett. **50**, 917 (1983).

[301] A. P. Young and M. Nauenberg, Phys. Rev. Lett. **54**, 2429 (1985).

[302] A. P. Young, J. D. Reger and K. Binder, in [34].

Figure Captions

1. Contours of $\chi_{SG} = 1$ (obtained via Monte Carlo simulations) for two-, three- and four-dimensional spin glasses in an external field compared to series expansions [272]. From ref. [110].

2. Waiting time experiments: Relaxation of the thermo-remanent magnetization for different waiting times t_w. The procedure mentioned in the text is sketched in the inset: the sample ($CdCr_{1.7}In_{0.3}S_4$) is cooled from above $T_g = 16.7K$ down to $0.6T_g = 10K$ in a 15 Oe field, kept in the field during t_w, then the field is cut off. From ref. [292]. .

3. Top: Decay of the remanent magnetization of $(Fe_xNi_{(1-x)})_{75}P_{16}B_6Al_3$ as function of time t for various temperatures ($T_g = 22.6K$) in a log-log plot. Bottom: Exponent m as a function of temperature obtained from a fit of the data of the upper figure to an algebraic decay $M_S(t) \propto t^{-m(T)}$ for $T \leq 0.97T_g$. From ref. [107].

4. The autocorrelation function $C(t, t_w)$ for the three-dimensional EA model with Gaussian couplings at $T = 0.5$ ($T_g \approx 0.9$ [23]). The system size is $L = 24$ and the data are averaged over 128 samples. From ref. [151], see [234] for similar data in case of the ±1-distribution.

5. The correlation length $\xi(t_w)$ of the two-dimensional EA-model with Gaussian couplings as a function of the waiting time t_w in a log-log plot. The straight lines are least-square fits toan algebraic growth law (13) with exponents $\alpha(T) = 0.044$, 0.066 and 0.081 for temperatures $T = 0.2$, 0.3 and 0.4, respectively. Data taken from ref. [237]. .

6. Phase diagram of the diluted antiferromagnet in a uniform field: $H = \sum_{\langle ij\rangle} \epsilon_i\epsilon_j S_i S_j - B\sum_i \epsilon_i S_i$ with 50% dilution ($\epsilon = 0$ or 1 with probability 1/2). PM means paramagnetic phase, AFM means antiferromagnetic long range order and SG is the intermediate spin glass phase (as explained in the text). From ref. [202]. .

7. Phase diagram of the diluted dipolar coupled Ising spin glass $LiHo_{0.167}Y_{0.833}F_4$ in the Γ–T plane. Filled circle follow from the dynamical behavior of the linear susceptibility [297], open circles from measurements of the nonlinear

susceptibility. We are interested in the region close to $T = 0$, $\Gamma = 1$. From ref. [298].

8. The dimensionless cumulant $g_{\rm av}$ for the (2+1)-dimensional classical model (31) for three different values of the coupling constant versus the system size L_τ in the imaginary time direction. The system size in the space direction is $L = 4$ ($\diamond$), 6 (+), 8 ($\Box$), 12 ($\times$) and 16 ($\triangle$). Since the maximum of $g_{\rm av}(L_\tau)$ is roughly independent of L at $K^{-1} \approx 3.30$ one concludes that the latter value is the critical coupling constant. From ref. [236]. .