Jamming and flocking in the restricted active Potts model

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We study the active Potts model with either site occupancy restriction or on-site repulsion to explore jamming and kinetic arrest in a flocking model. The incorporation of such volume exclusion features leads to a surprisingly rich variety of self-organized spatial patterns. While bands and lanes of moving particles commonly occur without or under weak volume exclusion, strong volume exclusion along with low temperature, high activity, and large particle density facilitates traffic jams. Through several phase diagrams, we identify the phase boundaries separating the jammed and free-flowing phases and study the transition between these phases which provide us with both qualitative and quantitative predictions of how jamming might be delayed or dissolved. We further formulate and analyze a hydrodynamic theory for the restricted APM with that predicts various features of the microscopic model.

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I. INTRODUCTION

Active matter systems are natural or artificially made, composed of large numbers of active particles, each of which consumes energy to self-propel or exert mechanical forces on the surroundings. The energy consumption from the surroundings happens at a single particle level where the interaction between individual particles occurs directly or through disturbances generated in the medium. In this process, the assembly of particles exhibits complex dynamics and a variety of collective motions. Collective motion leads to the spontaneous emergence of an orderly movement of large clusters of self-propelled individuals, called flocks, where the typical size of the clusters is significantly larger than the size of the individual [1–5].

Flocks are the spontaneous synchronized motion of large clusters that emerge for large densities and low noise. Flocking behavior plays a significant role in a wide range of systems across disciplines, including physics, biology, ecology, and neurosciences [6]. The spontaneous emergence of flocking transition is an out-of-equilibrium phenomenon that is abundant in nature. From human crowds, mammalian herds, bird flocks, and fish schools to unicellular organisms such as amoebae, bacteria, individual cells, and sub-cellular structures including cytoskeletal filaments, molecular motors, etc., all show remarkable active reorganization [7–10]. There is an intrinsic similarity between biological systems, which are internally driven and active matter systems; therefore, the physics of active matter has emerged as a valuable tool for understanding the principles behind the mechanics of biological systems.

A widely studied computational model mapping an active matter system is the Vicsek Model (VM) [11–15] where a simple numerical protocol was proposed analogous to ferromagnet demonstrating flocking transition for a system of particles from a high noise, low-density disordered phase to low noise, high density ordered phase. Here, the individual particle tends to align with its neighbors’ average direction of motion. Further studies on the flocking transitions were made thanks to the recent studies on the active Ising model (AIM) by Solon and collaborators [16–18] where they argued that the flocking transition can be seen as a liquid-gas phase transition rather than an order-disorder transition where the tuning parameters can dictate a transition from a disordered gaseous phase to ordered liquid phase with an intermediate liquid-gas coexistence region. Recently, a generalized active spin version of the AIM, the q-state active Potts model (APM) [19, 20] is studied on lattices where the ground state has q-fold degeneracy. With no restriction to the number of particles occupying a given lattice site, the APM shows a liquid-gas phase transition akin to the AIM. In addition, an exciting reorientation transition of the phase-separated profiles from transversal band motion to longitudinal lane formation is found, which was absent in the VM and the AIM. Lately, the q-state active clock model (ACM) [21, 22] is proposed as a natural discretization of the VM, which exhibits liquid-gas phase transition similar to the VM, AIM, and APM where the coexistence region shows a macrophase separation for small q values as in the AIM and APM and microphase separation for large q values as in the VM. The ACM [21] also manifests a reorientation transition for small q values as in the APM.

Now, jamming commonly occurs in a system of tiny
moving particles or large objects, or even living creatures gather in a place in significant numbers and tend to become immobile. One can think of various systems, ranging from sand piles and foams to traffic jams exhibiting such features [23–30]. Most familiar among all these is the traffic jam that generally arises when vehicles meet from the opposite. Studying the dynamics of autonomous agents navigating along a shared path is a pertinent problem for understanding automobile traffic [26–30], as well as collective navigation of animals [31, 32] and pedestrians [33–35]. Interestingly, jamming also arises when many agents travel in the same direction as efficiently as possible while avoiding collisions. As the density of the agent increases, directed motion and collision avoidance conflict, leading to collective congestion. Traffic jams can arise from different external as well as internal factors. External factors include poor road network, geometry and topology or poorly synchronized traffic regulation. However, factors such as mobility and intrinsic switching tendency also cause jamming and can motivate people to study how their effects can be altered.

The importance of intrinsic mobility or self-propulsion and seemingly benign interactions is well known in the field of active matter, where they may lead to events like motility-induced phase separation (MIPS) [36, 37] to completely novel spatial self-organization [10]. Many biological processes, such as embryogenesis [38], wound healing [39, 40], and cancer progression [38, 41], rely heavily on collective cell motility [42]. However, it is unclear which criteria govern the transition from freely moving single cells to an efficient collective motion. Different forms of living and artificially created active materials and swarms transition from a uniform fluid state to a phase-separated or clustered state [37, 43–45]. However, certain types of grouping can be counterproductive for tasks requiring a consistent flow. Traffic jams, competing bacterial biofilms, and robot swarms can cause high-density clogs that quickly stop moving and freeze at the jamming transition in confined active systems [43, 46]. It is critical in these systems to break up clusters and prevent them from developing.

In this paper, we have considered the $q$-state APM [19, 20] as our model system and sought to investigate the steady-state behavior of the $q = 4$-state APM when a volume exclusion effect restricts the occupancy of a given lattice site; in other words, the self-propulsion of an active particle to its neighboring site is a function of the maximum occupancy of the site. Earlier, a similar attempt was made by Peruani and collaborators [47] for a 4-state active system with a different particle flipping transition mechanism, where hopping to the nearest neighbor was possible only when the neighbor was empty. This led to a variety of self-organized spatial patterns such as traffic jams, gliders, and bands. Our work takes a systematic approach to the volume exclusion of self-propelled 4-state APM leading to specific patterns. In this approach, we set the following restrictions on the particle movement: (a) a lattice site can accommodate only a fixed number of particles (a purely repulsive hardcore restriction signifying complete volume exclusion beyond maximum capacity) and (b) a site can only accommodate a new particle probabilistically akin to the Boltzmann weight where the energy is replaced by a local field which is a function of the local particle density (a softcore restriction akin to the well known softcore Bose-Hubbard model). Physically, a more congested site would generate a stronger repulsive field that will function as a volume exclusion. In this context, we ask how the system behaves when subjected to these restrictions, the characteristics of different jamming phases, and the roles of the control parameters in determining the transitions between the different jammed and free-flowing phases.

Using numerical simulations, we obtain the following main results:

(a) When at most one particle is allowed per site (maximum site occupancy is one), a transition occurs from the disordered phase to MIPS [48–50] phase (phase separation into a dense and dilute phase due to the excluded volume repulsion) as we increase particle density and activity.

(b) An increase in temperature either makes the jamming less congested or helps overcome the jamming phase.

(c) An increase in the particle self-propulsion velocity or activity enhances the possibility of a jamming phase.

(d) Large average particle density facilitates a transition to the jammed phase. This paper is organized as follows. In Sec. II, we have defined the restricted active Potts model and provided details of the simulation protocols. Next, in Sec. III, we present numerical results of extensive Monte Carlo simulations of the microscopic model where simulation outcomes are discussed for three different restriction protocols, (i) the ‘maximum particle per site’ (MPS) is restricted to one akin to the active lattice gas (ALG) [51], (ii) hardcore restriction or MPS \( > 1 \), and (iii) softcore restriction or local repulsive field. In Sec. IV, we present a hydrodynamic description. We conclude this paper with a summary and discussion of the results in Sec. V.

II. MODELING AND SIMULATION DETAILS

We consider an ensemble of \( N \) particles defined on a two-dimensional square lattice of size \( L^2 \) with periodic boundary conditions applied on both sides, where \( L \) is the linear lattice dimension. The average particle density \( \rho_0 \) is then defined as \( \rho_0 = N/L^2 \). The model is built upon the APM [19, 20] in which the dynamics is governed either by the on-site flipping of the internal spin state or by nearest-neighbor hopping. Besides, we now propose restrictions on particle hopping. We suggest three types of mutually exclusive restrictions: a particle is allowed to hop to its neighbor if (a) that neighbor is empty or (b) the population of the neighboring site is less than the maximum occupation per site (hardcore restriction), or
\[
\Delta H_i = H_i^{\text{new}} - H_i^{\text{old}} = \frac{qJ}{\rho_i} (n_i^\sigma - n_i^\sigma' - 1) \tag{3}
\]

The flipping is then accepted with the rate:
\[
W_{\text{flip}}(\sigma \to \sigma') = \gamma \exp(-\beta \Delta H_i)
\]
\[
= \gamma \exp\left[ -\frac{q\beta J}{\rho_i} (n_i^\sigma - n_i^\sigma' - 1) \right], \tag{4}
\]
where \(\beta = 1/T\) is the inverse temperature. It should be noted that for \(\text{MPS} = 1\), only one particle is allowed per site and consequently, on-site alignment interaction is absent for this limit of the model. From Eq. (4), \(n_i^\sigma = 1\) and \(n_i^\sigma' = 0\) leads to \(\Delta H_i = 0\) and hence we have \(W_{\text{flip}}(\sigma \to \sigma') = \gamma\) as the flipping rate of particles for \(\text{MPS} = 1\). For hardcore and softcore restrictions, we take \(\gamma = 1\).

### B. Biased diffusion or hopping dynamics

The biased diffusion mechanism is similar to the process described in [19, 20]. A particle with state \(\sigma\) hops to a direction \(p\) with rate [19, 20]:
\[
W_{\text{hop}}(\sigma, p) = D \left(1 + \epsilon \frac{q\delta_{\sigma,p} - 1}{q - 1}\right), \tag{5}
\]
where \(\epsilon\) is the self-propulsion parameter. \(\epsilon = 0\) signifies pure diffusion and \(\epsilon = q - 1\) signifies complete self-propulsion.

Under the purely repulsive hardcore exclusion, biased diffusion is subjected to the maximum number of particles allowed per site set by the parameter \(\text{MPS}\) and Eq. (5) gets modified in the following way:
\[
W_{\text{hop}}^{HC}(\sigma, p) = W_{\text{hop}}(\sigma, p)\Theta(\text{MPS} - \rho_i)
\]
\[
= D \left(1 + \epsilon \frac{q\delta_{\sigma,p} - 1}{q - 1}\right) \Theta(\text{MPS} - \rho_i), \tag{6}
\]
where \(\Theta(\text{MPS} - \rho_i)\) is a Heaviside step function and is defined as follows:
\[
\Theta(\text{MPS} - \rho_i) = \begin{cases} 1, & \text{for } \rho_i < \text{MPS} \\ 0, & \text{otherwise} \end{cases}.
\]
\(\rho_i\) is the particle number at a neighboring site \(i\) to which a hopping is attempted. \(\text{MPS} = 1\) is a special case under the hardcore exclusion category where a move to the neighbor is only possible if that site is vacant. Therefore, unlike hardcore and softcore scenarios, on-site interactions between the particles are absent. This can also be thought of as an asymmetric simple exclusion process (ASEP) in which particles perform biased random walks under the hardcore repulsion that two particles cannot occupy the same site at a given time (\(\text{MPS} > 1\) is the non-lattice-gas variety of the ASEP). Therefore, \(\text{MPS} = 1\) is a special case under the hardcore exclusion category.
1 constructs the simple-exclusion “active lattice gas” [51] version of the APM. A softcore restriction would allow a particle hop to a neighboring site i from a randomly chosen site depending on the change in the local field. The local field is defined by:

\[ V(\rho_i) = U\rho_i(\rho_i - 1), \]

where \( U \) is an interaction coefficient that can be attractive (\( U < 0 \)) or repulsive (\( U > 0 \)). After hopping, the local field with \( \rho_i + 1 \) particles at site i becomes:

\[ V(\rho_i + 1) = U\rho_i(\rho_i + 1). \]

Particle hopping to site i is then accepted with probability:

\[ P = \min[1, \exp(-\beta\Delta V)] = \min[1, \exp(-2\beta U\rho_i)], \]

where \( \Delta V = V(\rho_i+1) - V(\rho_i) = 2U\rho_i \). Then, for softcore restriction, the modified form of Eq. (5) can be written as:

\[ W_{\text{hop}}^{SC}(\sigma, p) = W_{\text{hop}}(\sigma, p)\exp(-2\beta U\rho_i) - D \left(1 + \frac{\delta\sigma}{q} - 1\right)\exp(-2\beta U\rho_i). \]

(10)

\( U \) symbolizes the restriction strength that regulates particle accumulation on a single lattice site. Note that, a negative \( U \) would make \( \exp(-\beta\Delta V) > 1 \) (as \( \rho_i, \beta > 0 \)) which signifies unrestricted hopping or the field is attractive which helps particles to make a site more crowded. Therefore, \( U \leq 0 \) constitutes the limit of unrestricted APM [19] where particles can hop without constraint. In the current investigation, we always take \( U \) positive (a repulsive field that acts as volume exclusion).

C. Simulation Details

Simulation evolves in the unit of Monte Carlo steps (MCS) \( \Delta t \) resulting form a microscopic time \( \Delta t/N, N \) being the total number of particles. During \( \Delta t/N \), a randomly chosen particle either updates its spin state with probability \( p_{\text{flip}} = W_{\text{flip}}\Delta t \) or hops to one of the neighboring sites with probability \( p_{\text{hop}} = W_{\text{hop}}\Delta t \). For the q-state APM, an expression for \( \Delta t \) can be obtained by minimizing the probability of nothing happens \( p_{\text{wait}} = 1 - (p_{\text{hop}} + p_{\text{flip}}) \) [19],

\[ \Delta t = \left[qD + \exp(q\beta J)\right]^{-1}. \]

(11)

This hybrid Monte Carlo dynamics was used previously in the simulations of the AIM [16, 17], APM [19, 20], and ACM [21]. Instead of computing the \( \Delta t \) from the minimum of \( p_{\text{wait}} \) for systems that have small transition probabilities and therefore large \( p_{\text{wait}} \) (one has to generate random numbers until the chosen transition is accepted), one can also apply a Gillespie-like algorithm where one computes the time at which the next event will take place in the system.

III. NUMERICAL RESULTS

In this section, we present the numerical simulation results of the \( q = 4 \) restricted APM with MPS = 1, hardcore and softcore restrictions. The models are simulated on a square lattice of linear size \( L = 100 \) with periodic boundary conditions, where individual particle states \( \sigma = \{1, 2, 3, 4\} \) correspond to the movement directions right, up, left, and down, respectively. Simulations are performed for various control parameters: \( D = 1 \) is kept constant throughout the simulations, \( \beta = 1/T \) regulates the noise in the system, \( \rho_0 = N/L^2 \) defines the average particle density, and self-propulsion parameter \( \epsilon \) dictates the effective velocity the particles. Starting from a homogeneous initial condition, the Monte Carlo algorithm (Sec. II C) evolves the system under various control parameters until the stationary distribution is reached. Following this, measurements are carried out and thermally averaged data are recorded.

A. MPS = 1 (ALG version of the restricted APM)

In this segment, we present the results for \( q = 4 \) state APM with MPS = 1. Following [51], we define \( \gamma \) as the parameter that regulates the flipping where the flipping probability reads \( \gamma\Delta t \). We can then define the Péclet number \( Pe \) as the following:

\[ Pe = \frac{v}{\sqrt{D\gamma}}, \]

(12)

where \( v = 4D\epsilon/3 \) is the self-propulsion velocity in the hydrodynamic limit of the 4-state APM [19]. Now from Eq. (12), we get \( Pe = (4\epsilon/3)\sqrt{D/\gamma} \) for the lattice gas version of the 4-state APM. We notice that \( Pe \) is proportional to the self-propulsion parameter \( \epsilon \). Therefore, for small \( Pe \), diffusion dominates, and the effect of self-propulsion becomes negligible. The effect of self-propulsion activity becomes significant as \( Pe \) increases.

In Fig. 2, we demonstrate the formation of a diagonally jammed band typical of this model. Initially, the active particles nucleate stable clusters (where domains of all the four states can be visible) and eventually coalesce at later times \( (t = 10^5) \) into a diagonally jammed bulk phase that stabilizes in a steady state, much like nucleation and spinodal decomposition in liquid-gas phase coexistence. A careful examination of this diagonal jamming pattern reveals that the right (upper) and left (lower) domain boundaries are formed by multiple opposite spin states (e.g. for a diagonal band spanning from the bottom-right corner to the top-left corner, the right domain boundary is always formed by particles with \( \sigma = 3 \) and 4 and the left domain boundary is formed by particles of \( \sigma = 1 \) and 2). A two-state variant of this model having \( \sigma = 1 \) and 3 would result in a vertically jammed band [51] and a combination of \( \sigma = 2 \) and 4 would result in a horizontally jammed band. Therefore, diagonal jamming in 4-state APM arises when the steady-
FIG. 2: (color online) Evolution snapshots of the restricted APM with MPS = 1 demonstrating the formation of a diagonally jammed band. The right domain boundary is due to particles with $\sigma = 3$ (blue) and 4 (black) while the left domain boundary is due to particles with $\sigma = 1$ (red) and 2 (green). Parameters: $\text{Pe} = 50$ and $\rho_0 = 0.3$.

State culminates into orthogonally directed clusters intercepted by oppositely directed clusters as exhibited in Fig. 2.

The behavior of the 4-state APM with MPS = 1 is illustrated in Fig. 3 by representative late-stage snapshots as a function of propulsion strength $\text{Pe}$ and average particle density $\rho_0$. The system undergoes a transition from a disordered gaseous phase (at small densities and Pe) to MIPS [48–50] phase at both intermediate (large densities) and large Pe (small densities) via diagonal jammed bands at intermediate densities. MIPS refers to the phase separation into a high-density gas-like phase observed in a system of self-propelled particles (SPP) interacting via excluded volume repulsion. The physics of MIPS can be understood as slowing down of SPPs due to enhanced crowding when the local density of SPPs increases in some part of the system due to fluctuation, as shown in Fig. 3. The high-density phases (e.g. $\rho_0 > 0.5$) in Fig. 3 are disordered gas where particle motion is arrested, forming jammed clusters with a void in the middle. Cluster boundaries are created by two types of oppositely moving particles that can no longer occupy the preferred site due to maximum occupancy. Due to fluctuations along the domain boundaries, we observe diagonal bulk phase-separated jamming bands which we discussed in detail in the context of Fig. 2. Moreover, jamming width increases with $\rho_0$ and for a fixed $\rho_0$, system noise decreases with increasing Pe due to larger activity. Note that jammed clusters persist longer due to reduced flipping at larger Pe. In Ref. [47], a similar model was reported exhibiting traffic jams, gliders, and bands which we do not observe with our model due to the absence of ferromagnetic alignment interaction with the neighbors (we only have hardcore interactions).

In Fig. 4, we quantify the phase diagrams of the MPS = 1 model. Fig. 4(a) shows the quantitative analog of the diagram shown in Fig. 3 where different shades show the three phases (Gas, MIPS, and diagonal jamming). At large Pe, as the average density increases, the system transitions from a MIPS (I) cluster phase to a MIPS (II) jamming phase via a jammed diagonal band. The system remains in the gaseous phase at a small Péclet number because of low activity (diffusion dominates self-propulsion) and then transitions to the MIPS (II) jammed phase at large densities. We describe these phases as MIPS because here self-propelled particles with purely repulsive interactions use their motility to spontaneously separate into coexisting dense and dilute phases (see Fig. 3) and the speed of the motile particles decreases sufficiently as their local density increases leading to a phase-separated state in which the dense phase is of substantially reduced motility (a jammed phase). Since flipping probability decreases with activity [see Eq. (12)], particles tend to migrate in their favored directions and in that process promote the MIPS phase due to the imposed restriction.

We further compute the binodals $\rho_{\text{gas}}$ and $\rho_{\text{liq}}$ and plot the resulting phase diagram in Fig. 4(b). The binodals are the coexisting densities and physically signify the average densities of the gas and ordered phases at a given Pe. The binodals are estimated by calculating the average densities in different square boxes inside the high and low-density regions. From the diagram, the critical Pe is estimated as $\text{Pe}_c \simeq 8$ above which phase separation proceeds via spinodal decomposition. The shape of the phase diagram and the qualitative nature of the coexistence lines are similar to the diagrams obtained for
the active lattice gas [51] and the active Brownian particles [52].

B. Hardcore Restriction (MPS > 1)

This section presents our findings of the 4-state APM with hardcore restrictions where we restrict the maximum number of particles on a site (MPS > 1). A lower MPS signifies higher restriction on particle movement. 

Fig. 5(a) and Fig. 5(c) respectively show steady-state snapshots as a function of temperature and MPS for small ($\epsilon = 0.9$) and large ($\epsilon = 2.7$) propulsion velocities. In Fig. 5(a), the system exhibits jammed MIPS bands with well-defined domain boundaries for strong restriction whereas shows features of unrestricted APM as relaxation on particle movement increases. The jammed bands (MPS = 6) are almost immobile because the preferred directions of motion of the particles on the boundaries are inaccessible due to the small hardcore cutoff. The system manifesting a liquid phase at low temperature and a coexistence region (with transversely moving liquid band) at a relatively higher temperature (MPS = 12) is typical of unrestricted APM in the phase diagram: a gaseous phase at high $T$, an ordered liquid phase at low $T$, and a coexistence region at intermediate $T$ [19] where the band motion is transverse. At low MPS, the constraint prevents collective motion, resulting in a jammed phase for a large range of temperatures. As $T$ increases, the jammed region shrinks. High-temperature systems remain gaseous.

Fig. 5(d) shows the $T$ − MPS phase diagram for $\epsilon = 2.7$. The coexistence region is characterized by longitudinally moving liquid bands at intermediate temperatures, and the jammed phase occurs even at very high temperatures for small MPS. As portrayed in both the phase diagrams, fluctuations play a crucial role in the transition of the jammed phase to the coexistence or liquid phase by enhancing the probability of flipping. For $\epsilon = 0.9$, the hopping rate to non-preferred directions is substantial compared to $\epsilon = 2.7$. Thus, moderate self-propulsion and thermal fluctuations help break the jammed configuration more efficiently. Another difference is the MPS range. Particles hop quickly at large $\epsilon$. At higher MPS values, tending toward unrestricted APM, the liquid band of the coexistence region becomes narrower and more populated with increasing $\epsilon$. Unrestricted APM’s $\epsilon - \rho_0$ phase diagram shows this. As the liquid binodal value increases with $\epsilon$ (at a fixed $T$), so does the cutoff MPS.

Fig. 6(a) and Fig. 6(c) show system morphology as a function of average particle density $\rho_0$ and $T$ for (a) $\epsilon = 0.9$ (MPS = 15) and (b) $\epsilon = 2.4$ (MPS = 60). At low density, Fig. 6(a) shows a transition from coexistence to disordered gas as $T$ increases where the band in the coexistence region is moving transversely as $\epsilon$ is small. For $\rho_0 = 7$, the transition from liquid to disordered gas is observed as $T$ increases. No jamming occurs as $\text{MPS} = 15$ merely restricts particle hopping at this low density. At a larger density ($\rho_0 = 13$), jamming happens for all values of $T$ as larger $\rho_0$ requires higher MPS to avoid jamming. The jamming observed for large density shows that a high-density homogeneous MIPS phase is contained than the low $T$ cluster and we get a jammed morphology like MIPS without locally ordered distinct domains. As restriction is relaxed (MPS = 51), enhanced flipping at high temperature dissolves the jam observed at low temperature leading to a coexistence region. During this process, the system also exhibits the reorientation transition, a novel feature of the flocking phenomenon in the unrestricted APM [19], where we observe the particles in the band move in the transverse direction for $\epsilon = 0.9$ [blue band in Fig. 5(a)] whereas a longitudinally moving band for $\epsilon = 2.7$ [green band in Fig. 5(c)].
FIG. 5: (color online) (a & c) Steady-state snapshots of the 4-state restricted APM in the $T$–MPS plane for $\rho_0 = 4$. (a) For $\epsilon = 0.9$, an increase in MPS induces a transition from the jammed phase to a liquid phase at lower $T$ and to a coexistence region with transversely propagating liquid band at a higher $T$. (c) Snapshots for $\epsilon = 2.7$ where a jam-to-band transition occurs at large MPS and higher $T$. Here the coexistence band is longitudinal. Legend: red ($\sigma = 1$): right; green ($\sigma = 2$): up; blue ($\sigma = 3$): left; black ($\sigma = 4$): down; white: empty. (b & d) $T$–MPS phase diagrams for $\rho_0 = 4$. (b) For $\epsilon = 0.9$, an increase in the MPS transforms the system from a jammed to an ordered liquid phase at low $T$, and a coexistence region at moderate $T$. At high $T$, a gaseous phase is observed irrespective of MPS. (d) For $\epsilon = 2.7$, the jammed region is significantly larger than (b) and the system makes a transition to the liquid region at low $T$ and large MPS. As $T$ is increased gradually (for large MPS), liquid-to-gas transition occurs via a coexistence region at large MPS.

within the four well-defined liquid boundaries: a lower domain boundary of $\sigma = 2$, an upper domain boundary of $\sigma = 4$, a right domain boundary of $\sigma = 1$ and a left domain boundary of $\sigma = 3$. The high-density MIPS phase is a precursor to jams. MIPS clusters have mixed-state particles distributed homogeneously. Jammed phases are clusters with locally ordered domains of opposing states [47]. MIPS and jammed phases are distinguishable from cluster domains. Jammed phases are more stable than MIPS, which can cause jamming with proper control.

Small hopping probability to non-preferred directions for $\epsilon = 2.4$ causes jamming at $\text{MPS} = 60$, shown in Fig. 6(c). $T$ and MPS determine the fate of jammed clusters. At small $T$, jamming occurs even for small $\rho_0 = 7$ as particles move fast. Note that at a small $\epsilon = 0.9$, we observed a liquid phase for these values of $T$ and $\rho_0$. Increased temperature dissolves the jammed phase, allowing a longitudinally moving band as observed at $T = 1.42$ ($\beta = 0.7$) and $T = 2$ ($\beta = 0.5$). The jammed phase reappears at $T = 1.42$ for higher densities ($\rho_0 = 13$). A jammed state can be avoided at these densities by increasing the temperature. Enhanced flipping at higher $T$ changes jamming morphology from locally ordered to spatially homogeneous.

The $T$–$\rho_0$ phase diagrams of the RAPM with hardcore restriction are shown in Fig. 6(b) and Fig. 6(d). At low temperatures, Fig. 6(b) shows that the system mimics unrestricted APM behavior and transitions from a gaseous to an ordered liquid phase via liquid-gas coexistence. At higher densities, increased temperature causes jamming. At an intermediate $\rho_0$, increasing $T$ causes a coexistence phase from a liquid phase. Large densities require large MPS to prevent jamming; $\text{MPS} = 15$ is, therefore, insufficient for these clusters to avoid jamming. At high enough temperatures, the system is always gaseous, unaffected by MIPS and $\rho_0$.

For $\epsilon = 2.4$ and $\text{MPS} = 60$, we observe a jammed phase for $\rho_0 > 4$ at low $T$ values in Fig. 6(d). Increasing $T$ causes a coexisting phase for small $\rho_0$, but at large $\rho_0$, the system is liquid. A large $\epsilon$ allows particles to self-propel in the biased direction, which is unaltered at low temperatures. Particles of different states meeting at a point stay stuck for a long time, causing a jammed phase at low $T$. Temperature increase partially dissolves this situation for small densities. At large densities, $\text{MPS} = 60$ is sufficient for these clusters to form a liquid phase. Similar to Fig. 6(b), at high temperatures the system becomes gaseous regardless of $\rho_0$ and MIPS.

Fig. 7(a) shows steady-state snapshots and the corresponding phase diagram of the 4-state APM in $\epsilon - \rho_0$ plane for $\text{MPS} = 20$ and $\beta = 0.7$. For small particle velocity, the system exhibits a gaseous phase at a small density and an ordered liquid phase at a large density due to a relatively large MPS. At large $\epsilon$, although jamming is not observed for small densities (longitudinally moving coexistence band is observed), the system exhibits a traffic jam at a large density. Large $\epsilon$ promotes more particles accumulate at a lattice site. Since these particles have a very small probability to hop toward the non-biased directions, a higher MPS is required to avoid a jammed phase. The jammed cluster in Fig. 7(a) for $\rho_0 = 5$ and $\epsilon = 2.4$ is an example of the densest congestion. The appearance of such a cluster happens at low $T$ and large $\epsilon$ due to a lack of fluctuations and minimal hopping probabilities along non-preferred directions. A high particle density also enhances the possibility of such a traffic jam. Particles belonging to each of the ordered domains in such a dense, locally ordered cluster are unlikely to flip and break the dense congestion when $T$ is low or $\rho_0$ is high.

Fig. 7(b) represents the $\epsilon - \rho_0$ phase diagram of the RAPM with hardcore restriction for the parameters of Fig. 7(a). When $\epsilon$ is small, and $\rho_0$ is significantly lower than the $\text{MPS} = 20$, the system behaves like the unre-
FIG. 6: (color online) (a & c) Steady-state snapshots in $T - \rho_0$ plane for control parameters: (a) MPS = 15 and $\epsilon = 0.9$ and (c) MPS = 60 and $\epsilon = 2.4$. (b & d) $T - \rho_0$ phase diagrams of the RAPM with hardcore restriction as a function of particle speed $\epsilon$, (b) $\epsilon = 0.9$ and (d) $\epsilon = 2.4$. In (b), the liquid phase appears for intermediate densities but transitions to a jammed phase at higher densities whereas in (d), at large activity, jamming appears early and liquid phase is only observed when the temperature is moderately high.

FIG. 7: (color online) (a) Steady-state snapshots and (b) $\epsilon - \rho_0$ phase diagram of the RAPM with hardcore restriction for fixed control parameters: $T = 1.42$ ($\beta = 0.7$) and MPS = 20. An increase in $\rho_0$ and $\epsilon$ drives the system towards the jammed phase. At a smaller $\rho_0$, the jammed phase transitions into a coexistence region characterized by a high-density longitudinal lane ($\epsilon = 2.4$). The phase diagram in (b) resembles the unrestricted APM [19, 20] at small and intermediate $\epsilon$ but shows a transition to the jammed phase at higher $\rho_0$ and $\epsilon$. The $\epsilon$ for which the reorientation transition happens (transverse to longitudinal) and the $\rho_0$ for which a liquid-gas phase transition occurs with $\epsilon = 0$, coincide with the unrestricted APM [19, 20].

Restricted APM [19, 20]. In this case, the qualitative shape of the phase-diagram resembles that of the unrestricted APM where the binodal $\rho_{\text{gas}}$ delimit the gas and coexistence regions and the binodal $\rho_{\text{liq}}$ acts as the phase boundary between the coexistence and liquid regimes. The reorientation transition, which is a novel feature of the APM and where the system exhibits a transverse band motion at small $\epsilon$ and a longitudinal lane motion at large $\epsilon$, is also observed. The conventional phase diagram however breaks down at large $\rho_0$ and $\epsilon$ where a jamming transition from the coexistence region and the liquid phase is observed. At $\epsilon = 0$, similar to the unrestricted APM, a direct liquid-gas phase transition is observed at $\rho_0 \simeq 3$.

Systematic variation of magnetization is a crucial indicator of symmetry breaking. The plots in Fig. 8 show the magnetization against different control parameters where among the four different magnetizations corresponding to four internal states, we calculate the maximal magnetization $m_{\text{max}}$ using Eq. 2. It can be seen from the plots that magnetization changes abruptly across the phase boundaries denoted by dashed vertical lines. A fully ordered liquid state is characterized by $m_{\text{max}} \simeq 1$ and $m_{\text{max}} \simeq 0$ signifies a disordered gaseous phase. Fig. 8(a) shows $m_{\text{max}}$ versus $T$ for $\epsilon = 0.9$, $\rho_0 = 4$, and MPS = 8. At low $T$, a small $m_{\text{max}}$ indicates a jammed phase (locally ordered domains marginally rise the $m_{\text{max}}$ to a nonzero value). As the temperature is increased a sharp jump in magnetization ($m_{\text{max}} \simeq 0.7$) around $T = T_1^* \simeq 1.58$ indicates a transition to the coexistence region. For $T > T_2^* \simeq 1.8$ the amplitude of $m_{\text{max}} \simeq 0$ indicates a fully disordered phase. In Fig. 8(b), $m_{\text{max}}$ is plotted against $\rho_0$ for fixed $\epsilon = 1.2$, $\beta = 0.65$ ($T = 1.54$), and MPS = 10. At small densities, $m_{\text{max}} \simeq 0$ indicates a gas phase and at large densities, particles do not get enough space to move freely at MPS = 10 and create a jammed phase as evident from the $m_{\text{max}}$ value. At intermediate densities, the magnetization profile indicates a coexistence region where $m_{\text{max}}$ increases with $\rho_0$. The sharp fall of $m_{\text{max}} \simeq 0.8$ in Fig. 8(c) signifies the transition from coexistence region to the jammed phase around velocity $\epsilon = \epsilon^* \simeq 2.2$. The jammed phase occurs at a relatively high $\epsilon$ in Fig. 8(c) because of the large MPS value. Fig. 8 also remarks that the transition to the jammed phase from the coexistence region which we call the jamming transition is a first-order transition as demonstrated by the discontinuous jump of the order parameter at the transition points.

One can also distinguish a jammed phase from a freeflowing phase by investigating the mean-square displacement (MSD) of the high-density clusters corresponding to these two phases or by calculating the number fluctuations of the system in these two phases [53]. A jammed phase can be characterized by strongly suppressed number fluctuations and an oscillatory (because some jammed clusters oscillate around their mean posi-
max

0.2

0.8

0

1

0

0.2

0.4

0.6

0.8

1

1 2 3 4 5

Coexistence 
Jammed

(b)

0

0.2

0.4

0.6

0.8

1

0.6 1.2 1.8 2.4

Coexistence 
Jammed

(c)

\( m_{\text{max}} \)

\( m_{\text{max}} \)

\( m_{\text{max}} \)

1 1.2 1.4 1.6 1.8 2 2.2

Jammed

Coexistence 
Gaseous

FIG. 8: (color online) Normalized maximal magnetization \( m_{\text{max}} \) is plotted against \( T \), \( \rho_0 \), and \( \epsilon \) in (a), (b), and (c) respectively. \( m_{\text{max}} \) of the coexistence region is larger than the jammed phase. For the disordered gas phase, \( m_{\text{max}} = 0 \) and \( m_{\text{max}} \approx 1 \) for the liquid phase.

In this section, we present the results of our investigation of the 4-state APM with softcore restriction where the restriction is induced by the parameter \( U \) \((U > 0)\), which physically signifies the repulsive on-site interaction between the particles during hopping.

Fig. 9(a) and Fig. 9(c) depict the late-stage coarsening of 4-state APM in the \( T - U \) plane for \( \rho_0 = 4 \) and (a) \( \epsilon = 0.9 \) and (c) \( \epsilon = 2.7 \). No jamming is observed for small \( U(= 0.06) \), and the system shows an ordered liquid phase of particles with \( \sigma = 1 \) at low \( T \) and a transversely moving band of \( \sigma = 1 \) at high \( T \) signifying the coexistence region. At large \( U(= 0.16) \), we see a locally ordered, high-density cluster for \( T = 1 \), but a higher temperature destroys this cluster and the system exhibits orientational disorder, while a group of particles self-segregates into disordered particle aggregates. As explained in the context of hardcore restriction, an increased particle flipping probability with \( T \) helps the system dissipate a jammed state. The heterogeneous, arrested, dense cluster at \( T = 1 \) is the typical morphology of an extreme traffic jam where the shape and internal structure of the cluster depend on \( q \). We find, e.g., a square/rhombus cluster with four locally ordered domains for \( q = 4 \), a hexagonal cluster with six locally ordered domains for \( q = 6 \) (data are not shown here). For \( q \rightarrow \infty \) (active XY) [21], one can assume that the jammed phase in two-dimension would be a MIPS cluster typical of repulsively interacting active Brownian particles (ABP) [54]. Fig. 9(c) is analogous to Fig. 9(a) but for a larger velocity. Due to high propulsion and suppressed particle motion along the non-preferred directions at \( \epsilon = 2.7 \), the liquid phase (observed with \( \epsilon = 0.9 \)) becomes jammed at a fixed \( U = 0.06 \) and \( T = 1 \). When the temperature is increased to \( T = 1.66 \) \((\beta = 0.6)\), due to the increase in thermal fluctuation, the jammed phase changes into a coexistence region with longitudinally moving liquid bands. At a comparatively large \( U \), the jam persists at both low and high temperatures.

Fig. 9(b) and Fig. 9(d) respectively present the \( T - U \) phase diagrams for \( \epsilon = 0.9 \) and \( \epsilon = 2.7 \) at \( \rho_0 = 4 \). In Fig. 9(b), as the density is relatively large and velocity is small, the system shows all three phases of the unrestricted APM as a function of \( T \) for small values of \( U \). There is a transition from the gaseous phase at low \( T \) to an ordered liquid phase at high \( T \) via a liquid-gas coexistence region at intermediate \( T \); no jamming is observed. An increase of \( U \), however, changes this scenario. We observe that the system transitions from the liquid and coexistence region to the jammed phase at larger \( U \) due to the surge in volume exclusion imposed by \( U \). At very high temperatures, the system remains in the gaseous phase. In Fig. 9(d), we show that due to the high velocity of the particles, the system exhibits jamming at low temperatures even at small \( U \) and the liquid phase is observed only for very small \( U \). At high \( T \), for small \( U \), the coexistence region is observed due to the interplay between increased flipping at enhanced hopping of particles. At large \( U \), the system always manifests a jammed phase, and at high \( T \), a gaseous phase is observed analogous to Fig. 9(b).

Fig. 10(a) and Fig. 10(c) show the representative snapshots of 4-state APM as a function of average particle density \( \rho_0 \) and temperature \( T \) for (a) \( U = 0.07 \) and \( \epsilon = 0.9 \) and (c) \( U = 0.02 \) and \( \epsilon = 2.7 \). Fig. 10(a) indicates that the system behaves like the unrestricted APM [19, 20] at high temperature \((T = 2)\), a gaseous phase at a density \( \rho_0 = 2 \), a liquid phase at \( \rho_0 = 13 \) via a coexistence region at intermediate \( \rho_0 = 7 \). Jammed clusters appear at a very low temperature, and their size increases with increasing density. At an intermediate temperature \((T = 1.6)\), a liquid phase forms at \( \rho_0 = 7 \) due to moderate flipping and freedom to hop in the non-preferred directions, and jams are observed only at very high densities. With the increase of particle velocity \( \epsilon = 2.7 \), at low temperature \((T = 1)\), we find in Fig. 10(c) dense locally ordered clusters of mixed states embedded in a low-density background. The clusters are jammed by domains of oppositely directed particles that face each other and cannot dissolve due to less flipping at low temperatures. An increase in \( \rho_0 \) widens the jammed domain. At high temperature, \( T = 2 \) \((\beta = 0.5)\), the system is in the gaseous state at a small \( \rho_0 = 2 \). It transitions from the gaseous phase to the jammed region as density increases(unrestricted APM behavior). As the magnitude of \( U \) is not large \((U = 0.02)\), jamming is not observed for high temperatures.

The corresponding \( T - \rho_0 \) phase diagrams of the RAPM with softcore restriction are shown in Fig. 10(b) and Fig. 10(d). In both Fig. 10(b) and Fig. 10(d), we notice that the system replicates unrestricted APM behavior, with a transition from a gaseous to an ordered liquid phase occurring via a liquid-gas coexisting phase with increasing \( \rho_0 \) at intermediate to large \( T \). Temperature reduction, on the other hand, leads to an increase in the jammed region with increasing densities. Earlier stud-
ies have shown that reducing $T$ for a fixed $\rho_0$ leads to a liquid phase from a coexisting phase. As a result, at these high densities, the local density of clusters of individual states increases during coarsening to a liquid phase. However, softcore restrictions on SPPs are insufficient for these clusters to avoid jamming at these $\rho_0$ values because high densities necessitate small $U$. For sufficiently high temperatures, the system is always in a gaseous phase that is unaffected by $\rho_0$. The findings also indicate that a strong repulsive interaction ($U$) is required to generate a glass-like jammed state at low densities.

The steady-state snapshots as a function of $\epsilon$ and $\rho_0$ for $\beta = 0.7$ ($T = 1.42$) and $U = 0.07$ are shown in Fig. 11(a). At small velocity $\epsilon = 0.6$, we observe a gas phase at small density and a liquid phase of $\sigma = 1$ at large density. For large velocity, jams are seen as we increased density $\rho_0 = 5$, whereas, for the $\rho_0 = 2$, a transition from the gas phase to the coexistence region is observed.

The $\epsilon - \rho_0$ phase diagram of RAPM with softcore restriction is shown in Fig. 11(b). The system behaves nearly diffusively for very small $\epsilon$; therefore, jamming does not exist at small densities. However, with high particle propulsion, due to a significant restriction ($U = 0.07$), a jammed phase occurs even at small and intermediate densities where the cutoff $\epsilon$ value to jamming decreases with $\rho_0$. With the increase of $\epsilon$, transitions to the jammed phase occur faster for higher densities as more particles accumulate at a site with higher velocities. The $\epsilon - \rho_0$ phase diagram in Fig. 11(b) for $\epsilon \leq 1$ is analogous to the $\epsilon - \rho_0$ phase diagram of the unrestricted APM. At the zero activity limit ($\epsilon = 0$), the system behaves like the unrestricted APM at small $U$ where we see a direct liquid-gas transition with density whereas, at larger $U$, orientationally disordered self-segregated domains are observed instead of the ordered liquid phase (see Appendix E).

In Fig. 12(a), we plot maximal magnetization $m_{\text{max}}$ versus $T$ for $\epsilon = 0.9$, $\rho_0 = 4$, and $U = 0.14$ where $m_{\text{max}}$ is already defined in the context of Fig. 8. At low $T$, a nonzero but small value of $m_{\text{max}}$ characterizes the jammed phase. As the temperature is increased, thermal noise dissolves the jammed phase and a transition happens from the jammed phase to the coexistence region as indicated by the jump in the magnetization and the dashed vertical line at $T = T_1^* \approx 1.37$. For $T > T_2^* \approx 1.8$, we get $m_{\text{max}} = 0$ which indicates a disordered gas phase. Fig. 12(b) shows $m_{\text{max}}$ against $\epsilon$ for $\rho_0 = 7$, $\beta = 0.7$, and $U = 0.2$. At small velocities, due to slow propulsion and large density, the system exhibits a fully ordered liquid phase which is characterized by $m_{\text{max}} \approx 1$. At large velocities, $\epsilon > \epsilon^* \approx 0.75$, particles quickly accumulate locally and due to large $U = 0.2$, particle movements are heavily restricted and therefore we observe a jammed state with small $m_{\text{max}}$. In Fig. 12(c), variation of $m_{\text{max}}$ with $U$ is shown for $\beta = 0.95$ ($T = 1.05$), $\rho_0 = 4$, and $\epsilon = 0.9$. Small $U$ facilitates particle hopping which together with slow particles and a relatively large $\rho_0$ gives rise to the ordered liquid phase characterized by $m_{\text{max}} \approx 1$. For large $U$, however, one observes a jammed phase with small $m_{\text{max}}$ and this transition happens at $U = U^* \approx 0.12$. Similar to Fig. 8, the jamming transition with the softcore restriction is also first-order signifying that the specific origin of the restriction imposed on the particle movement does not alter the nature of the phase transition.

IV. HYDRODYNAMIC THEORY

In this section, we formulate the hydrodynamic continuum theory for the microscopic restricted APM. From the microscopic hopping and flipping rates, we derive the master equation for the probability density function $\rho_\sigma(\mathbf{x}; t)$ for a particle to be at the position $\mathbf{x}$ and in the state $\sigma$ at the time $t$. We only keep the first-order terms in the $|\mathbf{m}_i| \ll \rho_i$ expansion in the flipping rate (4). We obtain the hydrodynamic equations in Appendix A, for
observe a jammed phase adjoining both coexistence and liquid phases at large densities and low temperatures. These spinodal lines are shown in Fig. 4(b). For the Péclet number must be larger than $P_e = 4D\epsilon/\rho^2$, where $P_e = \epsilon/\mu$. Note for $U = 0$, this eigenvalue is always negative (since $\mu_0 < 0$ for $\rho_0 < \varphi_{\text{gas}}$). We

MPS = 1:

$$\partial_t \rho_\sigma = D_\parallel \partial_\parallel^2 \rho_\sigma + D_\perp \partial_\perp^2 \rho_\sigma - v \partial_\parallel [(1 - \rho)\rho_\sigma] - \gamma(4\rho_\sigma - \rho),$$

with $D_\parallel = D(1 + \epsilon/3)$, $D_\perp = D(1 - \epsilon/3)$, and $v = 4D\epsilon/3$. In Appendix B, we perform the linear stability analysis of the homogenous solution of the Eq. (13): $\rho_\sigma = \rho_0/4$. This leads to the spindals:

$$\varphi_\pm = \frac{3}{4} \pm \frac{\sqrt{D\gamma} + \rho_\sigma - \rho}{D\gamma},$$

where $\varphi_\pm$ is the Péclet number. Note that the Péclet number must be larger than $\varphi_{\text{gas}} = 8$, to observe MPS. These spinodal lines are shown in Fig. 4(b). For MPS $> 1$ one has to replace $1 - \rho$ by MPS $- \rho$ in the drift term on the r.h.s. of eq. (13).

In Appendix A, we also derive the hydrodynamic equa-
also perform the linear stability analysis of the ordered homogeneous solution of the Eq. (15) (supposed to be in state \( \sigma = 1 \), without loss of generality): \( \rho_1 = \rho_0(1 + 3M/4) \), and \( \rho_{2,3,4} = \rho_0(1 - M/4) \), with \( M \) satisfying the equation

\[
2\beta J(1 + M) - 1 - \frac{r}{\rho_0} - \alpha M^2 = 0. \tag{17}
\]

The perturbation along \( x \) is stable only if

\[
\lambda_{||} = -D(1 + \xi) + Q_1 \frac{D\epsilon}{3} + Q_3 \left( \frac{4D\epsilon}{3} \right)^2 \exp(-\xi) \tag{18}
\]

is negative and the perturbation along \( y \) is stable only if

\[
\lambda_{\perp} = -D(1 + \xi) - Q_1 \frac{D\epsilon}{3} + Q_2 \left( \frac{4D\epsilon}{3} \right)^2 \exp(-\xi) \tag{19}
\]

is negative, with the quantities

\[
Q_1 = \frac{\mu + \nu}{3\mu - \nu} - M\xi, \tag{20}
\]

\[
Q_2 = \frac{4\mu}{(3\mu - \nu)(3\kappa + \nu)} - \frac{(1 - M)\xi}{3\kappa + \nu}, \tag{21}
\]

\[
Q_3 = \frac{4\mu(-3\mu^2 + 2\mu\nu + \nu(4\kappa + \nu))}{(3\mu - \nu)^3(3\kappa + \nu)} - \frac{3\mu[\kappa(1 + 3M) + \mu(1 - M)] - \nu[\kappa(1 - M) + \mu(1 - 5M)]}{(3\mu - \nu)^3(3\kappa + \nu)} \xi, \tag{22}
\]

and with \( \xi = S\rho_0 \), \( \mu = M(4\beta J - 2\alpha M + \alpha M^2 - 1) \), \( \nu = M(4\beta J + 2\alpha M + 3\alpha M^2 - 3) \), and \( \kappa = M(-12\beta J - \alpha M^2 + 1) \). These eigenvalues allow the determination of the velocity for which the reorientation transition occurs, which is denoted \( \epsilon_\ast \). Fig. 14(d) shows a numerical estimation of \( \epsilon_\ast \), and for a temperature smaller than \( T_* = (1 - \sqrt{22}/8)^{-1} \), for which the ordered solution exists, \( \epsilon_\ast \) is an increasing function of \( U \). However, the existence of the jammed phase cannot be derived from this linear stability analysis. Fig. 13 shows the velocity-density phase diagram computed with the numerical solutions of the Eq. (15) obtained with FreeFEM++ [55], a software package based on the finite element method [56]. This phase diagram is similar to the one computed with numerical simulations shown in Fig. 11(b), with a jammed phase present at large velocities and densities.

V. SUMMARY AND DISCUSSION

In this paper, we have undertaken a systematic investigation of the effect of volume exclusions in terms of restricted hoppings of self-propelled particles in the 4-state APM [19]. The constraints can be classified into two categories: (a) hardcore repulsion, where the maximum occupancy of a lattice site is fixed and (b) soft-core repulsion, where particle mobility is controlled by a local field that depends on the particle density of the neighboring site to which the particle is allowed to hop. We then perform extensive numerical simulations of this restricted APM under four control parameters: the temperature \( T = \beta^{-1} \), average particle density \( \rho_0 \), particle self-propulsion velocity \( \epsilon \) and restriction parameters MPS (“Maximum Particle per Site”) for hardcore repulsion and potential \( U \) (\( U > 0 \)) for the softcore repulsion.

In the limit of MPS = 1, our model transforms to an active lattice gas model without any on-site alignment interaction and we observe a variety of self-organized patterns which show the coexistence between dilute and dense phases typical of MIPS [48–50]. At very small activity, however, the system remains in a gaseous phase at low densities and exhibits a homogeneous aggregate phase at higher densities. The emergence of such diverse jamming patterns results from the interplay between the local density, local orientation and particle speed although we do not observe traffic jams, gliders, or bands reported for a similar kind of model [47] due to the absence of nearest neighbor interaction. We further characterize the system by the liquid and gas binodals with \( Fe = 8 \) as the critical Péclet value, and the resulting MIPS phase diagram agrees qualitatively with the MIPS phase diagram obtained for the active lattice gas [51]. For hardcore and softcore repulsions, the APM local alignment interaction is present and apart from the three prevalent phases of the APM (disordered gas, polar liquid, and liquid-gas coexistence region), the system exhibits diverse self-organized patterns as functions of different control parameters and volume exclusion effects. These patterns include locally ordered high-density traffic jams, MIPS, high-density homogeneous aggregate phases bounded by well-defined domains, and self-segregated clusters. Through several phase diagrams, we provide a complete description of the system and
demonstrate its large-scale behavior where the jamming of particles plays a predominant role. We reach the following conclusions from the phase diagrams that connect to the jamming transitions in several other systems [10, 43, 46, 51, 53, 57–59]: (a) increasing average particle density leads to overcrowding, promoting a jammed phase, (b) increasing particle self-propulsion velocity also enhances the possibility of jamming due to small hopping rates in the non-preferred directions of motion, and (c) thermal fluctuation facilitates dissolving a jamming phase via more flipping of the active particles.

We further corroborate our numerical investigation with a hydrodynamic description of the RAPM. We derive the hydrodynamic equations accounting for the evolution of the density field for MPS = 1 and softcore restriction. With a linear stability analysis of the homogeneous solutions, we derive the equations for the spinodal lines (analytically exact for MPS=1, implicit equations for softcore restriction) and numerically compute the velocity at the reorientation transition $\epsilon^*$. We also compute the phase diagram for the softcore RAPM, presenting a jammed phase for large velocities and densities.

To summarize, we have shown a discretized flocking model that allows volume exclusion for controlling particle mobility and can produce a vast spectrum of self-organized patterns ranging from jammed and MIPS phases to collective motion. It is known that particle speed, temperature, and density all influence the local flux of particles and net particle flux rises as density and speed increase. However, in the present scenario, an increase in temperature rapidly changes the state of particles and reduces the effective flow. If the volume exclusion is significant, the flow of the particles is considerably suppressed, promoting a jammed state. We discover that boosting the volume exclusion strength (lowering MPS and elevating softcore potential $U$) or particle density can significantly reduce mobility due to cluster formation and local jamming. We also find that particle transport is generally increased when the thermal fluctuations increase but the system becomes effectively jammed at large activity.

A frozen or jammed state, for example, is a robust feature of the discrete model used in the present study that resembles naturally occurring systems. A similar relationship can be discovered in a variety of real-world scenarios, including gliding bacteria [60], animal groups [61], pedestrian traffic jams [33–35], high-density migratory cells [62], jammed herds [63], and robot swarms [64]. These systems can cause high-density clogs that quickly arrest into a frozen state [18, 44, 47, 65]. It would be fascinating to investigate whether physiologically relevant active matter, such as swimming organisms, have evolved strategies to avoid self-clogging [43, 46, 53].

From a future perspective, it will be interesting to study the effects of volume exclusion on the transport and jamming of active particles on disordered landscapes. Quenched disorders are abundant in all natural systems and are known to suppress local interactions. The altered composition may have an impact on the universal behavior of unrestricted systems, both in equilibrium and out-of-equilibrium conditions [66–68]. It is also known that active matter systems with random quenched disorder undergo activity-induced jamming [46, 69]. Preliminary results on restricted and unrestricted APM with quenched disorder reveal interesting emerging behaviors of the SPPs [70]. Another extension of our research would be to look into the influence of softcore constraints on active systems with continuous symmetry, such as the Vicsek model where initial investigation suggests an arrest of the flocking state with the emergence of MIPS jammed clusters.

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Appendix A: Derivation of hydrodynamic equations

In this section, we will derive the hydrodynamic equation for the $q$-state RAPM. In Ref. [20], we presented the detail hydrodynamic description of the unrestricted APM and their numerical solutions. In RAPM, as we have modified only the rule of hopping dynamics of the particles (keeping the flipping rule unchanged), we present here the derivation of the hydrodynamic equations only for the hopping term. To represent the different hopping restrictions, we introduce a function $f(n_i)$ where $i$ denotes the arrival site. The form of this function is $f(\rho) = 1 - \rho$ for MPS = 1 and $f(\rho) = \exp(-s\rho)$ for the softcore RAPM where $s = 2\beta U$. Note that we can consider $f(\rho) = 1 - s\rho$ for MPS = 1/s.

Considering only the hopping transition, the master equation writes

\[ G_{i+p} = G_i + a\partial_p G_i + \frac{a^2}{2} \partial_p^2 G_i + O(a^3), \]  

which leads to the expression

\[ G_{i-p}H_i - G_iH_{i+p} = -a\partial_p[H_i G_i] + \frac{a^2}{2} [H_i \partial_p^2 G_i - G_i \partial_p^2 H_i] + O(a^3). \]  

Knowing that the sum of the derivatives of any function $F$ are

\[ \sum_{p=1}^{q} \partial_p^2 F = \sum_{p=1}^{q} (e_p \cdot e_x)^2 \partial_x^2 F + 2 \sum_{p=1}^{q} (e_p \cdot e_x)(e_p \cdot e_y) \partial_x \partial_y F \]  

we finally get

\[ \sum_{p=1}^{q} [G_{i-p}H_i - G_iH_{i+p}] = \frac{qa^2}{4} [H_i \nabla^2 G_i - G_i \nabla^2 H_i] + O(a^3). \]  

The master equation becomes

\[ \partial_t n_i^q = \frac{qD}{4} \left( 1 - \frac{\epsilon}{q-1} \right) a^2 [f(n_i) \nabla^2 n_i^q - n_i^q \nabla^2 f(n_i)] + \frac{qD\epsilon}{q-1} \left\{ -a\partial_n [f(n_i)n_i^q] + \frac{a^2}{2} [f(n_i)\partial_n^2 n_i^q - n_i^q \partial_n f(n_i)] \right\}. \]  

We can decompose $a^2 \nabla^2 = \partial_\parallel^2 + \partial_\perp^2$ and $a\partial_n = \partial_\parallel$, and denote $\rho_\sigma = \langle n_i^q \rangle$ as well as $\rho = \langle \rho_i \rangle$, which leads to the expression

\[ \partial_t \rho_\sigma = D_\parallel \left[ f(\rho)\partial_\parallel^2 \rho_\sigma - \rho_\sigma \partial_\parallel^2 f(\rho) \right] + D_\perp \left[ f(\rho)\partial_\perp^2 \rho_\sigma - \rho_\sigma \partial_\perp^2 f(\rho) \right] - v\partial_\parallel [f(\rho)\rho_\sigma], \]  

with the coefficients

\[ D_\parallel = \frac{qD}{4} \left( 1 + \frac{\epsilon}{q-1} \right), \quad D_\parallel = \frac{qD}{4} \left( 1 - \frac{\epsilon}{q-1} \right) \quad \text{and} \quad v = \frac{qD\epsilon}{q-1}. \]
Note that we can decompose
\[ f(\rho) \partial_t^2 \rho_\sigma - \rho_\sigma \partial_t^2 f(\rho) = \partial_t \left[ f(\rho) \partial_t \rho_\sigma - \rho_\sigma \partial_t f(\rho) \right] \] (A12)
which yields the RAPM equation
\[ \partial_t \rho_\sigma = D_\parallel \partial_t^2 \rho_\sigma + D_\perp \partial_t^2 \rho_\sigma - v \partial_\parallel \rho_\sigma + \sum_{\sigma' \neq \sigma} \left[ \frac{4 \beta J}{\rho} (\rho_\sigma + \rho_{\sigma'}) - 1 - \frac{r}{\rho} - \alpha \frac{(\rho_\sigma - \rho_{\sigma'})^2}{\rho^2} \right] (\rho_\sigma - \rho_{\sigma'}) , \] (A17)
with \( \alpha = 8(\beta J)^2 (1 - 2\beta J/3) \).

For MPS = 1, \( f(\rho) = 1 - \rho \) one has \( \partial_t f(\rho) = -\partial_t \rho \) and the RAPM equation is
\[ \partial_t \rho_\sigma = D_\parallel \partial_t^2 \rho_\sigma + D_\perp \partial_t^2 \rho_\sigma - v \partial_\parallel \rho_\sigma + 2 \partial_\parallel \rho_\sigma - 2 \partial_\perp \rho_\sigma - \gamma (4 \rho_\sigma - \rho) . \] (A18)
by introducing the flipping term according to \( W_{\text{flip}} = \gamma \). Solving these hydrodynamic equations with FreeFEM++, we obtain unphysical solutions exhibiting regions with negative densities as well as densities larger than one. According to Ref. [51], and considering an anisotropic diffusive constant as for APM [20], we will consider the following equation for the case MPS=1:
\[ \partial_t \rho_\sigma = D_\parallel \partial_t^2 \rho_\sigma + D_\perp \partial_t^2 \rho_\sigma - v \partial_\parallel \rho_\sigma + \frac{1}{(1 - \rho) \rho_\sigma} \left[ -(1 - \rho) \partial_\parallel \rho_\sigma + \partial_\perp \rho_\sigma \right] - \gamma (4 \rho_\sigma - \rho) . \] (A19)

For softcore RAPM, \( f(\rho) = \exp(-S \rho) \) one has \( \partial_t f(\rho) = -S \partial_t \rho f(\rho) \) and the RAPM equation is
\[ \partial_t \rho_\sigma = D_\parallel \partial_t \left[ \exp(-S \rho) \left( \partial_\parallel \rho_\sigma + S \rho_\sigma \partial_\parallel \rho \right) + D_\perp \partial_\perp \left[ \exp(-S \rho) \left( \partial_\perp \rho_\sigma + S \rho_\sigma \partial_\perp \rho \right) \right] - v \partial_\parallel \left[ \exp(-S \rho) \rho_\sigma \right] \right] + \sum_{\sigma' \neq \sigma} \left[ \frac{4 \beta J}{\rho} (\rho_\sigma + \rho_{\sigma'}) - 1 - \frac{r}{\rho} - \alpha \frac{(\rho_\sigma - \rho_{\sigma'})^2}{\rho^2} \right] (\rho_\sigma - \rho_{\sigma'}) . \] (A20)

**Appendix B: Linear stability analysis for \( q = 4 \)-state RAPM with MPS = 1**

The hydrodynamic equation is
\[ \partial_t \rho_\sigma = D_\parallel \partial_t^2 \rho_\sigma + D_\perp \partial_t^2 \rho_\sigma - v \partial_\parallel \rho_\sigma + \frac{1}{(1 - \rho) \rho_\sigma} \left[ -(1 - \rho) \partial_\parallel \rho_\sigma + \partial_\perp \rho_\sigma \right] - \gamma (4 \rho_\sigma - \rho) . \] (B1)

Using dimensionless coordinates \( \tau = \gamma t \) and \( \mathbf{X} = \sqrt{\gamma/D} \mathbf{x} \), the hydrodynamic rewrites
\[ \partial_\tau \rho_\sigma = D_\parallel \partial_\tau^2 \rho_\sigma + D_\perp \partial_\tau^2 \rho_\sigma - \text{Pe} \partial_\parallel \rho_\sigma + \frac{1}{(1 - \rho) \rho_\sigma} \left[ -(1 - \rho) \partial_\parallel \rho_\sigma + \partial_\perp \rho_\sigma \right] - (4 \rho_\sigma - \rho) . \] (B2)
with \( D_\parallel = 1 + \epsilon/3, \ D_\perp = 1 - \epsilon/3 \) and the Péclet number
\[ \text{Pe} = \frac{4 \epsilon}{3} \sqrt{\frac{D}{\gamma}} . \] (B3)
The only homogeneous solution is $\rho_\sigma = \rho_0/4$ for all states $\sigma$. We consider then a linear stability analysis for $\rho_\sigma = \rho_0/4 + \delta \rho_\sigma$, where $\delta \rho_\sigma \ll \rho_0$ is a small perturbation. Keeping only the first order terms in $\delta \rho_\sigma$, the hydrodynamic equation becomes

$$\partial_t \delta \rho_\sigma = D_\parallel \partial_{\parallel}^2 \delta \rho_\sigma + D_\perp \partial_{\perp}^2 \delta \rho_\sigma - \text{Pe}(1 - \rho_0) \partial_{\parallel} \delta \rho_\sigma + \text{Pe} \frac{\rho_0}{4} \partial_{\parallel} \delta \rho - (4 \delta \rho_\sigma - \delta \rho),$$

with $\delta \rho = \sum_\sigma \delta \rho_\sigma$. With simplifications we get

$$\partial_t \delta \rho_\sigma = \left( D_\parallel \partial_{\parallel}^2 + D_\perp \partial_{\perp}^2 \right) - \text{Pe} \left( 1 - \frac{5 \rho_0}{4} \right) \partial_{\parallel} - 3 \delta \rho_\sigma + \left[ \text{Pe} \frac{\rho_0}{4} \partial_{\parallel} + 1 \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}, \tag{B5}$$

Performing a Fourier transform in space, we obtain

$$\partial_t \delta \rho_\sigma = \left( -D_\parallel k_\parallel^2 - D_\perp k_\perp^2 \right) + i \text{Pe} \left( 1 - \frac{5 \rho_0}{4} \right) k_\parallel - 3 \delta \rho_\sigma + \left[ -i \text{Pe} \frac{\rho_0}{4} k_\parallel + 1 \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}, \tag{B6}$$

or in a simpler form

$$\partial_t \delta \rho_\sigma = \alpha(k_\parallel, k_\perp) \delta \rho_\sigma + \beta(k_\parallel) \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}, \tag{B7}$$

with

$$\alpha(k_\parallel, k_\perp) = (-D_\parallel k_\parallel^2 - D_\perp k_\perp^2 + i \text{Pe} \left( 1 - \frac{5 \rho_0}{4} \right) k_\parallel - 3, \tag{B8}$$

$$\beta(k_\parallel) = -i \text{Pe} \frac{\rho_0}{4} k_\parallel + 1. \tag{B9}$$

The stability of the homogeneous is then given by the eigenvalues of the matrix

$$M = \left( \begin{array}{cccc}
\alpha(k_x, k_y) & \beta(k_x) & \beta(k_y) & \beta(k_x) \\
\beta(k_y) & \alpha(k_y, -k_x) & \beta(k_y) & \beta(k_x) \\
\beta(-k_x) & \beta(-k_x) & \alpha(-k_x, -k_y) & \beta(-k_x) \\
\beta(-k_y) & \beta(-k_y) & \beta(-k_y) & \alpha(-k_x, k_y)
\end{array} \right). \tag{B10}$$

With the help of Mathematica [71], we get that 3 eigenvalues are always negative and the fourth eigenvalue writes

$$\lambda = \frac{1}{8} \left[ -4(D_\parallel + D_\perp) + \text{Pe}^2 (1 - \rho_0)(2\rho_0 - 1) \right] (k_x^2 + k_y^2) + O(k_x^4, k_y^4). \tag{B11}$$

The homogeneous solution is then stable if and only if

$$(1 - \rho_0)(2\rho_0 - 1) < \frac{8}{\text{Pe}^2}, \tag{B12}$$

leading to the spinodals $\varphi_\pm$ defined by

$$\varphi_\pm = \frac{3}{4} \pm \sqrt{\frac{\text{Pe}^2 - 64}{4\text{Pe}}}, \tag{B13}$$

and a critical Péclet Pe$_c = 8$, to observe the MIPS.

**Appendix C: Linear stability analysis for $q = 4$-state softcore RAPM**

The hydrodynamic equation is

$$\partial_t \rho_\sigma = D_\parallel \partial_{\parallel} \left[ \exp(-S\rho) \left( \partial_{\parallel} \rho_\sigma + S\rho_\sigma \partial_{\parallel} \rho \right) \right] + D_\perp \partial_{\perp} \left[ \exp(-S\rho) \left( \partial_{\perp} \rho_\sigma + S\rho_\sigma \partial_{\perp} \rho \right) \right] - \nu \partial_{\parallel} \left[ \exp(-S\rho) \rho_\sigma \right]$$

$$+ \sum_{\sigma' \neq \sigma} \left[ 4\beta J \rho_\sigma (\rho_\sigma + \rho_{\sigma'}) - \frac{1}{\rho} - \alpha \left( \frac{\rho_\sigma - \rho_{\sigma'}}{\rho^2} \right)^2 \right] (\rho_\sigma - \rho_{\sigma'}), \tag{C1}$$
with \( \alpha = 8(\beta J)^2(1 - 2\beta J/3) \). Note that for \( S = 0 \), we recover the APM hydrodynamic equation, with the same flipping term:

\[
I_{\text{flip}}(\sigma, \sigma') = \left[ \frac{4\beta J}{\rho} (\rho_{\sigma} + \rho_{\sigma'}) - 1 - \frac{r}{\rho} - \frac{\alpha (\rho_{\sigma} - \rho_{\sigma'})^2}{\rho^2} \right] (\rho_{\sigma} - \rho_{\sigma'}). \tag{C2}
\]

The homogeneous solutions are given by \( I_{\text{flip}}(\sigma, \sigma') = 0 \), and are then the same as for APM. The disordered homogeneous solution is \( \rho_{\sigma} = \rho_0/4 \), and the ordered homogeneous solution (supposed along state \( \sigma = 1 \)) is \( \rho_1 = \rho_0(1 + 3M)/4 \) and \( \rho_{2,3,4} = \rho_0(1 - M)/4 \) with the magnetization \( M \) following the equation:

\[
2\beta J(1 + M) - 1 - \frac{r}{\rho_0} - \alpha M^2 = 0, \tag{C3}
\]

or \( M = M_0 \pm M_1 \delta \) with \( M_0 = \beta J/\alpha \), \( M_1 = \sqrt{r/\alpha \rho_*} \) and \( \delta = \sqrt{(\rho_0 - \rho_*)/\rho_0} \), where \( \rho_* \) defined by

\[
\rho_* = \frac{8(1 - 2\beta J/3)r}{1 + 8(2\beta J - 1)(1 - 2\beta J/3)}, \tag{C4}
\]

is the critical density below which the ordered homogeneous solution does not exist, for a temperature below \( T_c = (1 - \sqrt{72}/8)^{-1} \approx 2.417 \) [19].

**Linear stability analysis for the disordered homogeneous solution.** We take \( \rho_{\sigma} = \rho_0/4 + \delta \rho_{\sigma} \) and \( \rho = \rho_0 + \delta \rho \), with \( \delta \rho = \sum_{\sigma} \delta \rho_{\sigma} \). The restrictive term becomes \( f(\rho) \approx \exp(-S\rho_0)(1 - S\delta \rho) \). The hopping term writes

\[
I_{\text{hop}} \simeq D_\parallel \exp(-S\rho_0) \left( \partial_\parallel \delta \rho_{\sigma} + \frac{S\rho_0}{4} \partial_\parallel \delta \rho \right) + D_\perp \exp(-S\rho_0) \left( \partial_\perp \delta \rho_\sigma + \frac{S\rho_0}{4} \partial_\perp \delta \rho \right) - v \exp(-S\rho_0) \left( \partial_\parallel \delta \rho_{\sigma} - \frac{S\rho_0}{4} \partial_\parallel \delta \rho \right). \tag{C5}
\]

Merging \( \sigma \) and \( \sigma' \neq \sigma \) terms, we get

\[
I_{\text{hop}} \simeq \exp(-S\rho_0) \left[ \left( 1 + \frac{S\rho_0}{4} \right) (D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2) - v \left( 1 - \frac{S\rho_0}{4} \right) \partial_\parallel \right] \delta \rho_{\sigma}
+ \exp(-S\rho_0) \frac{S\rho_0}{4} \left[ D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2 + v\partial_\parallel \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}. \tag{C6}
\]

The flipping term writes

\[
I_{\text{flip}}(\sigma, \sigma') \simeq \mu_0 (\delta \rho_{\sigma} - \delta \rho_{\sigma'}), \tag{C7}
\]

with \( \mu_0 = 2\beta J - 1 - r/\rho_0 \). Then, in the Fourier space, the hydrodynamic equation becomes

\[
\partial_t \delta \rho_{\sigma} = \left[ A(k_\parallel, k_\perp) + 3\mu_0 \right] \delta \rho_{\sigma} + \left[ B(k_\parallel, k_\perp) - \mu_0 \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}, \tag{C8}
\]

with

\[
A(k_\parallel, k_\perp) = \exp(-S\rho_0) \left[ \left( 1 + \frac{S\rho_0}{4} \right) (-D_\parallel k_\parallel^2 - D_\perp k_\perp^2) + i k_\parallel v \left( 1 - \frac{S\rho_0}{4} \right) \right] \tag{C9}
\]

\[
B(k_\parallel, k_\perp) = \exp(-S\rho_0) \frac{S\rho_0}{4} \left[ -D_\parallel k_\parallel^2 - D_\perp k_\perp^2 - i k_\parallel v \right]. \tag{C10}
\]

The stability matrix is

\[
M_{\text{gas}} = \begin{pmatrix}
A(k_x, k_y) + 3\mu_0 & B(k_x, k_y) - \mu_0 & B(k_x, k_y) - \mu_0 & B(k_x, k_y) - \mu_0 \\
B(k_y, -k_x) - \mu_0 & A(k_y, -k_x) + 3\mu_0 & B(k_y, -k_x) - \mu_0 & B(k_y, -k_x) - \mu_0 \\
B(-k_x, -k_y) - \mu_0 & B(-k_x, -k_y) - \mu_0 & A(-k_x, -k_y) + 3\mu_0 & B(-k_x, -k_y) - \mu_0 \\
B(-k_y, k_x) - \mu_0 & B(-k_y, k_x) - \mu_0 & B(-k_y, k_x) - \mu_0 & A(-k_y, k_x) + 3\mu_0
\end{pmatrix}. \tag{C11}
\]
Supposing $k_x = k$ and $k_y = 0$ (since no preferred direction), up to the order $O(k^3)$, the eigenvalues of this matrix are

$$
\lambda_{gas}^1 = 4\mu_0 - D_\perp \exp(-S\rho_0)k_2^2,
$$

$$
\lambda_{gas}^2 = \exp(-S\rho_0) \left[ -(1 + S\rho_0) D_\parallel + D_\perp \frac{2}{2} + (1 - S\rho_0) \exp(-S\rho_0) \frac{v^2}{8\mu_0} \right] k^2,
$$

$$
\lambda_{gas}^{3,4} = 4\mu_0 \pm \frac{\sqrt{2k_2}}{2} \exp(-S\rho_0) - \exp(-S\rho_0) \left[ 3D_\parallel + D_\perp \frac{4}{4} + (1 - S\rho_0) \exp(-S\rho_0) \frac{v^2}{16\mu_0} \right] k^2.
$$

The disordered homogeneous solution is then stable if $\mu_0 < 0$ and

$$
\lambda_{gas} = -(1 + \xi)D + (1 - \xi) \exp(-\xi) \frac{v^2}{8\mu_0},
$$

with $\xi = S\rho_0$ is negative. In Fig. 14(a), we have represented the velocity-density stability diagram for $\beta = 0.75$ and $U = 0.5$ according to the sign of this eigenvalue. Note that the last inequality is always fulfilled only when $U = 0$, meaning that $\mu_0 < 0$ only impacts the spinodals: $\varphi_{gas}(U = 0) = r/(2\beta J - 1)$, independent of $\epsilon$. However, we cannot extract an analytical expression for the spinodal $\varphi_{gas}$ for all $U$ values.

FIG. 14: (color online) (a) and (b) Velocity-density stability diagram for the disordered and ordered homogeneous solution for $\beta = 0.75$ and $U = 0.5$. (a) The stability region of the disordered solution is plotted in blue, according to the eigenvalue $\lambda_{gas}$ given by Eq. (C15). (b) The stability region of the ordered solution is plotted in blue, according to the eigenvalues $\lambda_\parallel$ and $\lambda_\perp$ given by Eqs. (C39) and (C40), respectively. The ordered solution is only unstable under longitudinal perturbations ($\lambda_\parallel > 0$) in the light blue region, and under transverse perturbations ($\lambda_\perp > 0$) in the light red region. (c) $\epsilon_\ast$ value as a function of the temperature $T$, for several potential $U$. (d) $\epsilon_\ast$ value as a function of the temperature $T$ and the potential $U$. $\epsilon_\ast \geq 3$ means no reorientation transition, and only transverse bands are observed in the coexistence region.
Linear stability analysis for the ordered homogeneous solution. We consider the ordered solution along the right state, and we take \( \rho_1 = \rho_0(1 + 3M)/4 + \delta \rho_1, \rho_{2,3,4} = \rho_0(1 + 3M)/4 + \delta \rho_{2,3,4} \) and \( \rho = \rho_0 + \delta \rho \), with \( \delta \rho = \sum_\sigma \delta \rho_\sigma \). The restrictive term becomes \( f(\rho) \approx \exp( -S \rho_0 ) (1 - S \delta \rho ) \). The hopping term of the right state \( \sigma = 1 \) writes

\[
I_{\text{hop}}^{(1)} \approx D_\parallel \exp( -S \rho_0 ) \frac{S \rho_0}{4} (1 + 3M) \partial_\parallel \delta \rho + D_\perp \exp( -S \rho_0 ) \partial_\perp \frac{S \rho_0}{4} (1 + 3M) \partial_\perp \delta \rho
\]

\[\quad - v \exp( -S \rho_0 ) \partial_\parallel \delta \rho_\sigma - \frac{S \rho_0}{4} (1 + 3M) \partial_\parallel \delta \rho_\sigma \].

(C16)

Merging \( \sigma \) and \( \sigma' \neq \sigma \) terms, we get

\[
I_{\text{hop}}^{(1)} \approx \exp( -S \rho_0 ) \left\{ 1 + \frac{S \rho_0}{4} (1 + 3M) \right\} (D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2) - v \left[ 1 - \frac{S \rho_0}{4} (1 + 3M) \right] \partial_\parallel \delta \rho_\sigma
\]

\[\quad + \exp( -S \rho_0 ) \frac{S \rho_0}{4} (1 + 3M) \left[ D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2 + v \partial_\parallel \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}. \]

(C17)

The hopping term of the other states \( \sigma \neq 1 \) writes

\[
I_{\text{hop}}^{(2)} \approx \exp( -S \rho_0 ) \left\{ 1 + \frac{S \rho_0}{4} (1 - M) \right\} (D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2) - v \left[ 1 - \frac{S \rho_0}{4} (1 - M) \right] \partial_\parallel \delta \rho_\sigma
\]

\[\quad + \exp( -S \rho_0 ) \frac{S \rho_0}{4} (1 - M) \left[ D_\parallel \partial_\parallel^2 + D_\perp \partial_\perp^2 + v \partial_\parallel \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}. \]

(C18)

where the coefficients \( 1 + 3M \) have been replaced by \( 1 - M \). We may note for \( M = 0 \), we recover the expression of \( I_{\text{hop}} \), given by Eq. (C6), calculated for the disordered solution.

The flipping terms imply the right state write

\[
I_{\text{flip}}(1, \sigma') \approx M \left\{ (4 \beta J - 2 \alpha M) \delta \rho_1 + (4 \beta J + 2 \alpha M) \delta \rho_{\sigma'} + \left[ 2 \beta J (1 + M) - \frac{r}{\rho_0} - 2 \alpha M^2 \right] \delta \rho \right\}.
\]

(C19)

Using Eq. (C3), we get

\[
I_{\text{flip}}(1, \sigma') \approx M \left[ (4 \beta J - 2 \alpha M) \delta \rho_1 + (4 \beta J + 2 \alpha M) \delta \rho_{\sigma'} + (\alpha M^2 - 1) \delta \rho \right] \equiv \gamma_1 \delta \rho_1 + \gamma_2 \delta \rho_{\sigma'} + \gamma_3 \delta \rho.
\]

(C20)

The flipping terms which does not imply the right state write

\[
I_{\text{flip}}(\sigma, \sigma') \approx M (\alpha M - 4 \beta J) (\delta \rho_\sigma - \delta \rho_{\sigma'}) \equiv \gamma_4 (\delta \rho_\sigma - \delta \rho_{\sigma'}). \]

(C21)

Then we have the terms:

\[
I_\sigma = \sum_{\sigma' \neq \sigma} I_{\text{flip}}(\sigma, \sigma') \approx \begin{cases} 
3(\gamma_1 + \gamma_3) \delta \rho_\sigma + (\gamma_2 + 3 \gamma_3) \sum_{\sigma' \neq \sigma} \rho_{\sigma'}, & \text{if } \sigma = 1, \\
-(\gamma_1 + \gamma_3) \delta \rho_1 + (-\gamma_2 - \gamma_3 + 2 \gamma_4) \delta \rho_\sigma - (\gamma_3 + \gamma_4) \sum_{\sigma' \neq [1, \sigma]} \rho_{\sigma'}, & \text{if } \sigma \neq 1.
\end{cases}
\]

(C22)

Then, in the Fourier space, the hydrodynamic equation becomes

\[
\partial_t \delta \rho_\sigma = \begin{cases} 
\left[ A_1(k_\parallel, k_\perp) + 3 \mu \right] \delta \rho_\sigma + \left[ B_1(k_\parallel, k_\perp) + v \right] \sum_{\sigma' \neq \sigma} \delta \rho_{\sigma'}, & \text{if } \sigma = 1, \\
\left[ A_2(k_\parallel, k_\perp) + \kappa \right] \delta \rho_\sigma + \left[ B_2(k_\parallel, k_\perp) - \mu \right] \delta \rho_1 + \left[ B_2(k_\parallel, k_\perp) - \frac{\kappa + v}{2} \right] \sum_{\sigma' \neq [1, \sigma]} \delta \rho_{\sigma'}, & \text{if } \sigma \neq 1.
\end{cases}
\]

(C23)
with
\[
A_1(k_{||}, k_\perp) = \exp(-S\rho_0) \left\{ \left[ 1 + \frac{S\rho_0}{4} (1 + 3M) \right] \left( -D_{||} k_{||}^2 - D_{\perp} k_{\perp}^2 \right) + i k_{||} v \left[ 1 - \frac{S\rho_0}{4} (1 + 3M) \right] \right\},
\]
\[
B_1(k_{||}, k_\perp) = \exp(-S\rho_0) \frac{S\rho_0}{4} (1 + 3M) \left[ -D_{||} k_{||}^2 - D_{\perp} k_{\perp}^2 - i k_{||} v \right],
\]
\[
A_2(k_{||}, k_\perp) = \exp(-S\rho_0) \left\{ \left[ 1 + \frac{S\rho_0}{4} (1 - M) \right] \left( -D_{||} k_{||}^2 - D_{\perp} k_{\perp}^2 \right) + i k_{||} v \left[ 1 - \frac{S\rho_0}{4} (1 - M) \right] \right\},
\]
\[
B_2(k_{||}, k_\perp) = \exp(-S\rho_0) \frac{S\rho_0}{4} (1 - M) \left[ -D_{||} k_{||}^2 - D_{\perp} k_{\perp}^2 - i k_{||} v \right],
\]
\[
\mu = \gamma_1 + \gamma_2 = M(4\beta J - 2\alpha M + \alpha M^2 - 1),
\]
\[
\nu = \gamma_2 + 3\gamma_3 = M(4\beta J + 2\alpha M + 3\alpha M^2 - 3),
\]
\[
\kappa = -\gamma_2 - \gamma_3 + 2\gamma_4 = M(-12\beta J - \alpha M^2 + 1).
\]

The stability matrix is
\[
M_{\text{lin}} = \begin{pmatrix}
A_1(k_x, k_y) + 3\mu & B_1(k_x, k_y) + \nu & B_1(k_x, k_y) + \nu & B_1(k_x, k_y) + \nu \\
B_2(k_y, -k_x) - \mu & A_2(k_y, -k_x) + \kappa & B_2(k_y, -k_x) - (\kappa + \nu)/2 & B_2(k_y, -k_x) - (\kappa + \nu)/2 \\
B_2(-k_x, -k_y) - \mu & B_2(-k_x, -k_y) - (\kappa + \nu)/2 & A_2(-k_x, -k_y) + \kappa & B_2(-k_x, -k_y) - (\kappa + \nu)/2 \\
B_2(-k_y, k_x) - \mu & B_2(-k_y, k_x) - (\kappa + \nu)/2 & B_2(-k_y, k_x) - (\kappa + \nu)/2 & A_2(-k_y, k_x) + \kappa
\end{pmatrix}
\]
\[
\lambda_{\text{lin},x}^2 = \frac{3\kappa + \nu}{2} - D_{||} k_x^2,
\]
\[
\lambda_{\text{lin},y}^2 = \frac{3\kappa + \nu}{2} - D_{\perp} k_y^2,
\]
\[
\lambda_{\text{lin},xy}^2 = \frac{\mu + \nu}{3\mu - \nu} + M\xi - \frac{2D_{||} + 2D_{\perp}}{3(\mu - \nu)} k_x k_y - \frac{9D_{||}\mu - (2\mu - 2\nu)\nu}{3(\mu - \nu)} k_x^2 - \frac{9D_{\perp}\mu - (2\mu - 2\nu)\nu}{3(\mu - \nu)} k_y^2.
\]

Now we look at a perturbation in the $y$ direction ($k_x = 0$). Up to the order $O(\kappa^4)$, the eigenvalues of this matrix are
\[
\lambda_{\text{lin},y}^{1,2} = \frac{3\kappa + \nu}{2} \pm \frac{1}{\sqrt{3}} i k_y v - \frac{2D_{||} + 2D_{\perp}}{3} \left[ 9(3\kappa + \nu)(\kappa - 2\mu + \nu) \right] k_y^2 + \frac{4\nu v^2}{9(3\mu - \nu)(\kappa - 2\mu + \nu)} k_y^2,
\]
\[
\lambda_{\text{lin},y}^3 = (3\mu - \nu) - \frac{9D_{||}\mu - (2\mu - 2\nu)\nu}{3(3\mu - \nu)} + \frac{4\nu v^2}{9(3\mu - \nu)(\kappa - 2\mu + \nu)} k_y^2,
\]
\[
\lambda_{\text{lin},y}^4 = \frac{-2D_{||} + 2D_{\perp}}{3\mu - \nu} - \frac{D_{||}(1 - M) + D_{\perp}(1 - M)}{2} \mu + \frac{4\nu v^2}{9(3\mu - \nu)(\kappa + \nu)} - \frac{1 - M}{3\kappa + \nu} v^2 k_y^2.
\]

The ordered homogeneous solution is then stable if $3\kappa + \nu < 0$ and $3\mu - \nu < 0$ for the two different perturbations. This result was already observed for APM [19], and allows the selection of the position magnetization solution: $M = M_0 + M_1 \delta$. However, the stability of the two different perturbations differs from $\lambda_{\text{lin},x}^4$ and $\lambda_{\text{lin},y}^4$. The perturbation
along $x$ is stable only if
\[
\lambda_{\parallel} = \frac{\text{Re} \, \lambda^t_{\text{ homo},x}}{k_T^2 \exp(-\xi)} = -D(1 + \xi) + \left[ \frac{\mu + \nu}{3\mu - \nu} - M\xi \right] \frac{D\epsilon}{3} - \left[ \frac{4\mu[-3\mu^2 + 2\mu\nu + \nu(4\kappa + \nu)]}{(3\mu - \nu)^3(3\kappa + \nu)} + \frac{3\mu\kappa(1 + 3M) + \mu(1 - M) - \nu\kappa(1 - M) + \mu(1 - 5M)}{(3\mu - \nu)^2(3\kappa + \nu)} \right] \left( \frac{4D\epsilon}{3} \right)^2 \exp(-\xi) \quad (C39)
\]
is negative and the perturbation along $y$ is stable only if
\[
\lambda_{\perp} = \frac{\text{Re} \, \lambda^t_{\text{ homo},y}}{k_T^2 \exp(-\xi)} = -D(1 + \xi) - \left[ \frac{\mu + \nu}{3\mu - \nu} - M\xi \right] \frac{D\epsilon}{3} + \left[ \frac{4\mu}{(3\mu - \nu)(3\kappa + \nu)} - \frac{(1 - M)\xi}{3\kappa + \nu} \right] \left( \frac{4D\epsilon}{3} \right)^2 \exp(-\xi) \quad (C40)
\]
is negative. We may note that these eigenvalues are those obtained in Ref. [19] for $\xi = 0$. In Fig. 14(b), we have represented the velocity-density stability diagram for $\beta = 0.75$ and $U = 0.5$ according to the sign of these two eigenvalues. Here, due to the dependence of $\xi$ on the density $\rho_0$, we have not derived an expression of $\epsilon_*$ for which the reorientation transition occurs, but we have computed a numerical estimation in Figs. 14(c) and 14(d). We may mention that for some intermediate temperature values (between 0 and $T_c$) and positive potential $U$, no reorientation transition occurs. This means only transverse band motion can be observed for these $(T, U)$ values. Moreover, for temperatures not close to $T_c$, $\epsilon_*$ is an increasing function of $U$.

**Appendix D: Mean-squared displacements (MSD) of particles in the jammed state**

Here, we explore the appearance of arrested states through measurements of the MSD of individual particles. The MSD of $N$ number of particles in the system at time $t$ (which quantifies how the particles move from their initial positions under various volume exclusion effects) is defined as
\[
R^2(t) = \frac{1}{N} \sum_{i=1}^{N} |r_i(t) - r_i(0)|^2, \quad (D1)
\]
where $r_i(t)$ is the instantaneous position of the $i$-th particle at time $t$. For ballistic motion, $R^2 \sim t^2$ while for diffusive motion $R^2 \sim t$. For an arrested or jammed state, however, MSD is proportional to $t^\nu$ where $x \sim 0$ [59].

In Fig. 15(a), we show $R^2$ versus $t$ (on a log-log scale) for unrestricted APM [19] as a function of $\beta$ ($T^{-1}$). At small $\beta$, which physically signifies the gaseous phase, the system obeys the diffusive growth $R^2 \sim t$ whereas, for $\beta = 1.1$, where the system exhibits the liquid phase, we observe two distinct regimes in the MSD. The small $t$ limit is characterized by a diffusive regime with $R^2 \sim t$ growth whereas a ballistic growth regime characterized by the power-law $R^2 \sim t^\nu$ is observed at large $t$. In the liquid phase, advective force (self-propulsion) plays a crucial role as the system exhibits the ballistic growth regime, which is a collision-free regime in which particles travel freely after a majority of the particles switch in the same state.

MSD for hardcore restriction as a function of $\beta$ are shown in Fig. 15(b). At small $\beta$, $\beta = 0.5$, the MSD shows a diffusive growth as the system is in the gas phase. For large $\beta$ (which signifies the liquid phase for the mentioned control parameters), we observe a crossover from the diffusive growth at small $t$ to a ballistic growth at large $t$ as observed in Fig. 15(a). For intermediate $\beta$, the small $t$ diffusive growth regime is followed by a plateau in the MSD at large $t$, which signifies a jammed or arrested state. A similar crossover in MSD can be seen for a random walker confined in a box, where the MSD crosses over from a diffusive growth ($R^2 \sim t$) to a plateau once the walker sticks to the walls [59].

The MSD plots signifying the jammed phase describe the fact that initially the particles (the system is initially prepared homogeneous) do not feel the effect of the hopping restrictions and move diffusively but as the system coarsens with time, particles feel the restricted environment and finally, the system reaches the steady state jammed phase.

**Appendix E: Diffusion limited ($\epsilon = 0$) phase due to softcore restriction**

In Fig. 16, we show the late-stage representative snapshots of the system in the $U - \rho_0$ plane for $\beta = 0.7$ at the zero velocity ($\epsilon = 0$) limit where particle movement is controlled by diffusion. For small $U$, the system behaves like the unrestricted APM [19] and with increasing density, we notice a direct gas-liquid phase transition without any
coexistence regime similar to the unrestricted APM. However, with enhanced restriction on particle movement through $U$, the system exhibits orientational disorder instead of a liquid phase at high densities, where particles self-segregate into small domains. This is because the restriction on particle hopping is greater with larger $\rho_0$ and $U$ preventing the system to transform into a liquid phase. The size of the self-segregated clusters diminishes with increasing $U$ restricting the particle movement further.
For small $U$, similar to the unrestricted APM, we observe a phase transition from a disordered gaseous phase to an ordered liquid phase as density is increased. At large $U$, particles can only segregate locally into small clusters forming a homogeneous disordered phase at large spatial scales.