Optimal number of agents in a collective search and when to launch them

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Search processes often involve multiple agents that collectively search a randomly located target. While increasing the number of agents usually decreases the time at which the first agent finds the target, it also requires resources to create and sustain more agents. In this letter, we consider collective search costs that do not only account for the search time but also for the costs associated with launching and sustaining agents. First a general formalism is presented for independent agents in terms of the target survival probability for a single-agent search, where agents are allowed to be launched at arbitrary times. With this, the optimal number of searchers to be launched simultaneously at the initial time is analytically derived. Then, the optimal times at which later agents should be launched are calculated for different classes of the single-agent survival probabilities demonstrating how short- and long-time behaviors impact the optimal strategies. Finally the costs for launching new searchers are compared with those for resetting a single one.

The term search processes encompasses any phenomenon in which the encounter of agents with a target is important. They include chemical reaction kinetic [1, 2], micro-organisms scavenging for food [3], immune cells searching for pathogens [4], animal foraging and hunting [5, 6], or swarming robots used in rescue missions [7, 8]. In most instances, search protocols need to be optimized in some way. Most frequently, the typical time to find the target needs to be minimized with respect to the parameters governing the search process. Recently, considerable efforts have been dedicated to identify and analyze efficient and optimal search strategies. For searches with a single agent, the efficiency of various types of stochastic processes have been investigated, e.g. intermittent or Lévy walks [9–14], stochastic resetting [15–19], or non-markovian searches [20–24].

More recently, collective search processes have also been studied, not only for non-interacting agents [25], but also for systems where searchers can interact and communicate [26–31]. To this day, the number of searchers has however very rarely been considered as a variable to be optimized. While it is clear that for nearly all search processes the mean search time decreases with the number of searchers, adding more searchers may have a non-negligible cost in terms of the required resources. For a human search problem where one needs to pay agents and material resources to perform the search, one can very easily understand that it is not optimal to hire as many agents as possibly available as it would have a huge financial cost. Similarly, in an immune response process, hiring more cells to find a pathogen requires a substantial amount of metabolic energy. This is precisely the motivation for the question that we are raising in this letter: given the cost associated with launching and sustaining an agent in a collective search process, what is the optimal number of them and when to launch them?

Survival probability: We consider an arbitrary stochastic search process in which N agents search for a single immobile target located at position $\mathbf{r}_T$. The i^{th} walker is introduced at time $t_i \geq t_{i-1}$ and position $\mathbf{r}_i$. The N-agent process terminates once one searcher reaches the target. The individual processes can be arbitrary, provided that the single-agent survival probability (SASP) of the target $s_i(T, \mathbf{r}_i, \mathbf{r}_T)$ associated with the searcher i, i.e. the probability that the searcher i has not found the target until time t, is well defined. Note that s_i could be different for each searcher if their dynamics is not identical. Because the searchers are independent, the overall survival probability of the target is given by $S(T, \{t_i, \mathbf{r}_i\}, \mathbf{r}_T) = \prod_{k=1}^{N} s_k(T - t_k, \mathbf{r}_k, \mathbf{r}_T)\Theta(T - t_k)$, where Θ is the Heaviside function. Defining $\varrho_T(\mathbf{r}_T)$ and $\varrho_S(\mathbf{r}_1, \cdots, \mathbf{r}_N)$ the probability distributions for the target position $\mathbf{r}_T$ and initial coordinates of each searcher $\mathbf{r}_1, ..., \mathbf{r}_N$ respectively, the averaged survival probability is defined as $\bar{S}(T, \{t_i\}) = \int d\mathbf{r}_T \int d\mathbf{r}_1 \cdots \int d\mathbf{r}_N \varrho_T(\mathbf{r}_T)\varrho_S(\{\mathbf{r}_i\})S(T, \{t_i, \mathbf{r}_i\}, \mathbf{r}_T)$.

In order to proceed we assume that 1) all individual survival probabilities are identical, 2) the launch positions are random (i.i.d.), 3) the target position is fixed or the search domain is finite and has periodic boundary conditions, 4) the survival probability depends only on the initial distance from the target, and 5) the launch positions are random (uniformly distributed). With these assumptions $\bar{S}$ can be written as

$$\bar{S}(T, \{t_i\}) = \prod_{k=1}^{N} \bar{s}(T - t_k)\Theta(T - t_k) \tag{1}$$

where $\bar{s}$ has a different definition depending on the last assumption (see SM, section 1, for details). Note that $\bar{s}$ in principle depends on the system size such that all the results presented in this paper depend on it through $\bar{s}$.

Search cost: In order to take into account the number of launched searchers and the required resources to define the search cost K, we construct it as a weighted sum of three contributions, namely

$$K = J_T T + J_S \sum_{i=1}^{N} (T - t_i)\Theta(T - t_i) + K_L \sum_{i=1}^{N} \Theta(T - t_i) \quad (2)$$

The first term weighted by the *target cost rate* J_T quantifies a cost associated with the presence of the target and can be interpreted as a rate of damage due to the presence of the target. The second term weighted by the *searcher sustaining rate* J_S quantifies the amount of resources required to sustain one searcher per unit time. Finally, the last term weighted by the *searcher launching cost* K_L quantifies the amount of resources required to launch a searcher. For compactness, we introduce the normalized parameters $\gamma = J_S/J_T$ and $\kappa = K_L/J_T$ and set $J_T = 1$ as our cost rate unit for the rest of the paper.

Optimizing a search strategy with respect to the cost function K, eq. (2), means to minimize it with respect to the launch times $\{t_i\}$. In SM, section 2, we show that the mean search cost can then be written as

$$\bar{K}(\{t_i\}) = \sum_{n=1}^{N} \left[\kappa \bar{S}\left(t_n\right) + (1 + n\gamma) \int_{t_n}^{t_{n+1}} \bar{S}\left(t\right) dt \right] \ , \quad (3)$$

We also introduce $\Delta_n = t_n - t_{n-1}$ to be the *launch interval* for the n^{th} searcher.

Uniqueness and simultaneous launching: As for any optimization problem, the first question to be investigated is to know whether there is a unique global minimum of the cost function. For two searchers, we show in section 3 of the Supplemental Material that a sufficient condition for the uniqueness of the minimum for the z-quantiles and the means of the search cost is for the SASP $s(t)$ and its derivative to be logarithmically convex, in which case the launch time minimizing an arbitrary quantile can be found analytically. Proving the uniqueness of a local local minimum is much more difficult for $N > 2$ in general and has to be investigated numerically, as it is done later in the manuscript. Assuming the absence of a local maximum for the mean search cost, we can derive a criterion to decide whether it is favorable to introduce searchers simultaneously or if one should launch them one after the other. We show in section 4 of the Supplemental Material that $\partial_{t_{i-1}} \bar{K} > \partial_{t_i} \bar{K}$ for all $i > 1$ if $t_i^* = t_{i-1}^*$. This implies that the optimal search strategy is such that there can not be searchers launched simultaneously later than at the start of the process. Moreover, the optimal number of agents N_{sim} introduced simultaneously at $t = 0$ is the largest integer k that verifies $\partial_{t_k} \bar{K}_N > 0$ for $t_1 = \cdots = t_k = 0$, under the constraint $\partial_{t_p} \bar{K}_N = 0$ for $p > k$. There we have $\partial_{t_k} \bar{K} = k^{-1} + (k - 1)\kappa s'(0)$ such that N_{sim} is

$$N_{sim} = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{2} - \frac{1}{\kappa s'(0)}} \right\rfloor \overset{|\kappa s'(0)| \to 0}{\simeq} \frac{1}{\sqrt{-\kappa s'(0)}} \quad (4)$$

Surprinsingly, N_{sim} does not depend on γ at all: no matter how much it costs to sustain a searcher, the number of agents to be introduced into the system at $t = 0$ will only be governed by the launching cost κ. In addition, if $s(t)$ is very sharply decreasing at short times – i.e. if the probability to find the target quickly is high – there is no interest in launching multiple searchers initially as the benefit in the search time would be overcompensated by the launching cost.

Test cases: While the statistical properties of the search cost depend on the details of the SASP, in a vast majority of single-agent search processes $s(t)$ has a similar functional form such that we can categorize them into different classes. At long times $s(t)$ either decays exponentially (e.g. in confined domains [32, 33]) or algebraically (e.g. in open space [34]). Faster decays are extremely rare. Note that we consider here only SASP whose probability to eventually find the target is 1, i.e. $\lim_{t\to\infty} s(t) = 0$. For short times, we consider three cases: (i) $s'(0) = r(0) = 0$, i.e. the search can not be infinitely fast, leading to non-monotonic – and non-convex – first-passage time distributions [35, 36] (ii) $0 < s'(0) < \infty$, i.e. the probability to find the target at arbitrarily short times is finite, a frequent feature of processes with random initial positions of the searchers [37] (iii) $s'(0) = -\infty$, i.e. the probability density to find the target at $t = 0$ diverges, as it is observed in some simple diffusive processes [38].

We therefore consider test cases which combine specific short and long time behaviors. To do this, we combine $f^{exp} : t \mapsto e^{-\lambda t}$ and $f^{alg} : t \mapsto (1 + \lambda\theta^{-1}t)^{-\theta}$ characterizing the long-time-behavior with either $g_0 : x \mapsto \sin(\pi x/2)$, $g_{fin} : x \mapsto 1$ or $g_{inf} : x \mapsto 2 \arcsin(x)/\pi$, referred to as the *flat*, *mild* and *sharp* cases respectively and characterizing the short-time behaviors. The six SASP we focus on are therefore of the form $s_{\alpha,\beta} = g_\alpha \circ f^\beta$, c.f. SM, section 5 for a visualization, and will be used to discriminate the influence of the short and long-time behavior of the SASP on the optimal launching strategies. To identify them, we performed gradient descent optimization, where we used $\theta = 2$ for the algebraic cases. We ran the minimization procedure for increasing values of N, where the largest value of N was taken as the lowest value of n for which $S(t_n) < 0.001$. Among the six test cases under study, four of them had unique local minima of the mean search cost, namely the *mild* and *sharp* cases. They correspond to log-convex SASP, which is consistent with the prediction for 2 searchers. No local maximum was found such that equation (4) holds for these cases. The two other cases, i.e. non-convex SASPs, lead to more complex structures for the mean search cost and are analyzed separately.

Convex SASP: The exponential case $s(t) = e^{-\lambda t}$ can be analyzed analytically, c.f. SM, section 6. In this case, the optimal strategy is to launch N_{sim} searchers at $t = 0$ and none later, where $s'(0) = -\lambda$ is used in equation

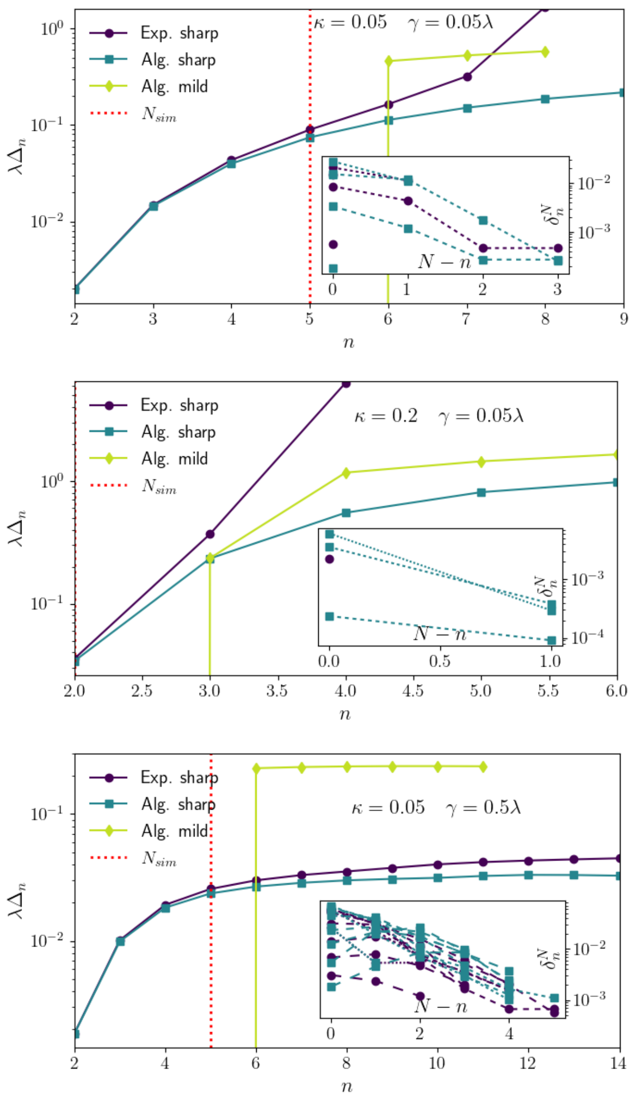

FIG. 1. Normalized launch intervals $\lambda\Delta_n$ as a function of the number of launched agents n for three convex test SASP and various values of κ and γ. the inset shows the relative variation δ_n^N as a function of $N-n$, for which $\Delta_n^{N+1} - \Delta_n^{N+1}$ was found larger than the tolerance of the gradient descent algorithm.

(4). This result is independent of the total number of available searchers N. For the three other convex cases, we first investigate to which extent the total number of available agents impacts the optimal introduction times by defining the relative variation $\delta_n^N = 1 - \Delta_n^{N+1}/\Delta_n^N$ upon having a new searcher in the reservoir, where the superscript N refers to the total number of searchers. As shown in the insets of Fig. 1, δ_n^N decays exponentially with $N-n$ starting with a relatively low amplitude. The optimal launch times t_n obtained in the limit $N \to \infty$ are therefore a good approximation of the actual optimal ones for a finite number N of available searchers.

In the limit $N \to \infty$, our results allow to identify the optimal launching strategies and to classify them, as shown in Fig. 1. We observe that the short-time behavior governs the launch time of the first agents while the long-time behavior governs the introduction of later agents. For the initially *mild* SASP, we have $N_{sim} \geq 1$ such that multiple walkers may be introduced simultaneously at the start of the process. For the *sharp* SASP, it holds $N_{sim} = 1$ such that one necessarily has to wait a certain time before launching a second walker. For searchers introduced at later times, the optimal launching intervals Δ_n converge to a constant value as $n \to \infty$ for algebraic SASP while they diverge for exponential ones. The number of agents around which this transition occurs decreases with κ and increases with γ and roughly corresponds to introduction times t_n such that $\lambda t_n \sim 1$. This is consistent with the observation that optimal launching intervals increase with κ and decrease with γ as shown in Fig. 2. While the dependence on κ is not surprising as a large launching cost should reward to wait longer before launching a new searcher, the dependence on γ is less intuitive. It is in fact preferable to launch searchers at a higher frequency when the *sustaining* rate γ is larger, indicating that the gain in the overall search time overcompensates the larger rate of resources required to sustain the new searchers. This trend holds for all tested SASP and appears to be a general result for a wide variety of search processes.

Non-convex case: For non-convex SASP multiple local minima may exist. For any $n \geq 2$ we have $\partial_{t_n}\bar{K} \geq 0$ for $t_2 = \cdots = t_n = 0$, the strategy consisting in launching n searchers simultaneously at $t = 0$ is *locally* optimal. Since the probability for the first searcher to find the target at very short times is low, there is no gain in waiting a short amount of time for launching next searchers compared to launching it together with the first one. However, this locally optimal strategy is not necessarily the globally best one, especially if the launching cost κ is high. We show in Fig. 3 an example of the optimal strategies for the test case with algebraic decay and $N = 4$. For low values of κ, there is no local minimum and the optimal strategy is to launch all searchers simultaneously. As one progressively increases κ, local minima with $t_k > 0$ for $k \geq n$ appear at $\kappa = \kappa_n^*$ and become the global minimum for $\kappa = \kappa_n^{**} > \kappa_n^*$.

Comparison with stochastic resetting: Instead of launching new searchers at the initial position in intervals one could also reset a single searcher to the origin at a certain rate, i.e. a search process with stochastic resetting, which has attracted a lot of interest recently [18, 39–42]. The fine-tuning of the resetting rates can have a significant impact on the overall search efficiency [43–46] and here we scrutinize when resetting is better than launching new searchers.

We adapt the main result of reference [16] and show that the mean search cost $\bar{K}_r$ is minimized for a fixed

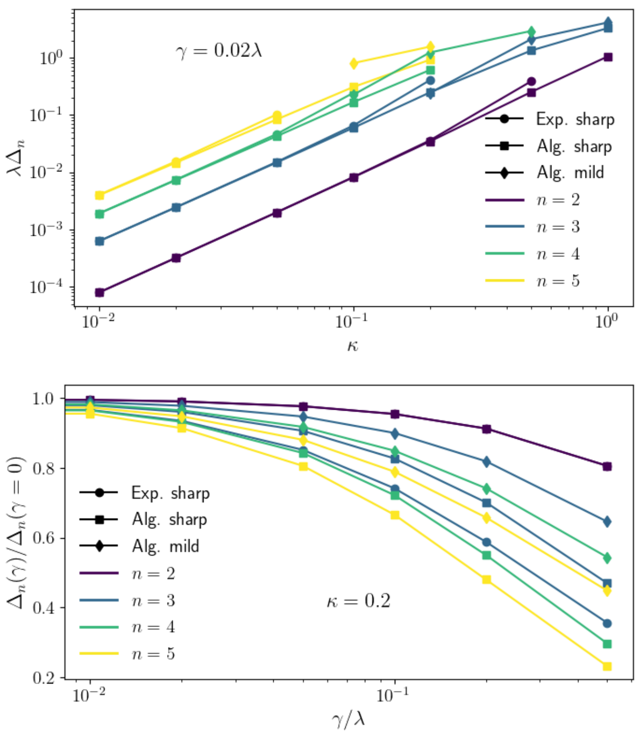

FIG. 2. Introduction intervals Δ_n as a function of κ (top) and γ (bottom) for three test cases. The values are normalized by Δ_n for $\gamma = 0$ in the bottom plot.

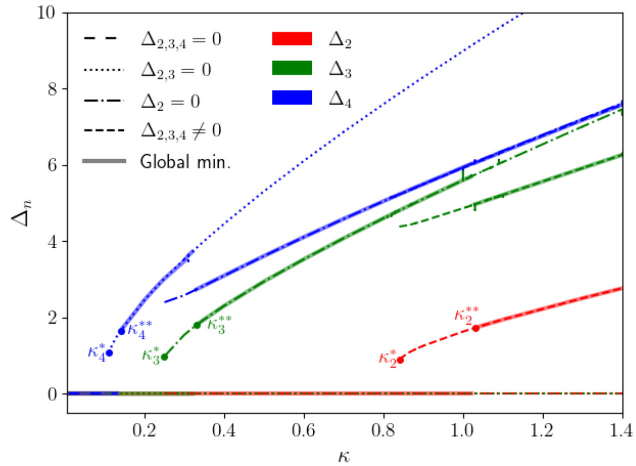

FIG. 3. Optimal introduction intervals Δ_n as a function of κ for $s(t) = s_{alg}^{flat}(t)$ with $N = 4$ and $\gamma = 0.01\lambda$. Different dashed line styles indicate different local minima while the solid line indicates the global one. Colors code for different searchers.

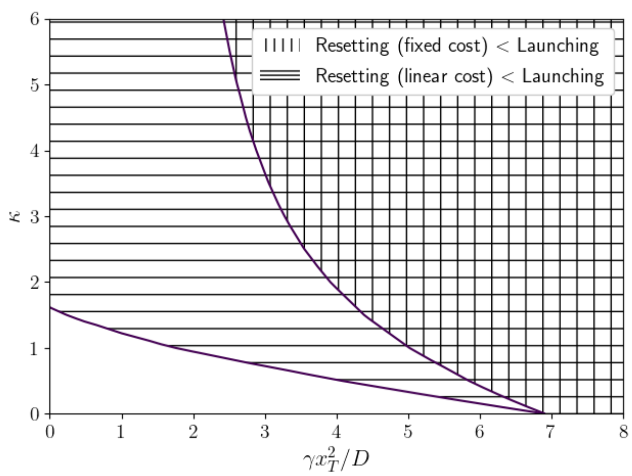

FIG. 4. Preferential strategy (resetting or launching new agents) for the 1-dimensional diffusive search in the (κ, γ)-plane, where we have used $\kappa = \kappa_r = f_r x_T$.

resetting time interval Δ, in which case it reads

$$\bar{K}_r(\Delta) = \left[(1+\gamma_r)G(\Delta) + \kappa_r(\Delta)\right]/F(\Delta) \tag{5}$$

where $F(\Delta) = 1 - s(\Delta)$ and $G(\Delta) = \int_0^\Delta s(\tau)d\tau$ (c.f. SM, section 7). Here κ_r is a resetting cost for which we consider two canonical cases: (i) a fixed cost κ_r and (ii) a mean resetting cost proportional to the distance between the searcher and its initial position upon resetting, $\kappa_r = f_r \cdot \delta x$, as used e.g. in [47, 48]. In this case the resulting search cost depends on the details of the single-agent search process and not only on its survival probability. Here we focus on a well-studied example, namely the one-dimensional diffusive search [49]. Each searcher is launched from $x = 0$ and diffuses freely with diffusion constant D. For a target placed at position $x_T > 0$ we have $s(t) = \mathrm{erf}(x_T/2\sqrt{Dt})$. For both cases (constant and linear resetting costs) the total search cost $\bar{K}_r$ can be calculated analytically (c.f. SM) and yields $\min_\Delta \bar{K}_r(\Delta)$ with $\min_\Delta \bar{K}_l(\Delta)$ can be compared with the cost for launching $\bar{K}_l$. Figure 4 shows when launching new searcher is better than resetting a single one: essentially as long as the costs for sustaining a searcher, γ multiplied with the diffusive timescale, x_T^2/D, is sufficiently small, provided the costs for creating new searchers, κ is also small enough. For linear resetting cost, there is even a value of κ above which resetting is always better than launching regardless of the value of γ. Note, however, that in practical applications it could be much harder, if not unfeasible, to reset searchers, in contrast to adding new searchers.

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Discussion and conclusion: In this letter, we have identified optimal strategies for when to launch new non-interacting agents in a search process and showed how the functional form of the single-agent survival probability impacts the overall optimal launching times of successive searchers. This is the first work to investigate in details how to optimize a search process considering not only the overall search time but also other costs relative to the resources required for the search. This was then compared to a canonical resetting process.

As a direction for future works, we make the following points. First, as mentioned early in the text, our results are valid under some assumptions on the single-agent process (mainly identical searchers and translational invariance). It is not clear yet whether our current results would still hold if these assumptions do not hold, e.g. for N-species models or for searches with obstacles.

Our work could also be adapted for non-Markovian searches, where the information gathered throughout the search by all walkers is shared and used to adjust optimal introduction times and locations, as it has for instance been studied for stochastic resetting processes [19, 50]. This would contribute to the current emerging fields of optimal collective searches in which one wants to optimize interaction and communication channels for agents to search together efficiently. We believe that this letter is a first step towards the formalization of such questions, which will be tackled in future studies.

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Optimal number of agents in a collective search and when to launch them
Supplemental Material

Hugues Meyer, Heiko Rieger

SURVIVAL PROBABILITY

Let us consider the joint survival probability defined as

$$S(T,\{t_i,\mathbf{r}_i\},\mathbf{r}_T)=\prod_{k=1}^{N}s_k(T-t_k,\mathbf{r}_k,\mathbf{r}_T)\Theta(T-t_k) \tag{1}$$

where s_k is the individual target survival probability for the searcher k. Let us now average it over the target position and searchers coordinates to find

$$\bar{S}(T,\{t_i\})=\int d\mathbf{r}_T\int d\mathbf{r}_1\cdots\int d\mathbf{r}_N\varrho_T(\mathbf{r}_T)\varrho_S(\{\mathbf{r}_i\})\prod_{k=1}^{N}s_k(T-t_k,\mathbf{r}_k,\mathbf{r}_T)\Theta(T-t_k) \tag{2}$$

where $\varrho_T(\mathbf{r}_T)$ and $\varrho_S(\mathbf{r}_1,\cdots,\mathbf{r}_N)$ are the probability distributions for the target position $\mathbf{r}_T$ and initial coordinates of each searcher $\mathbf{r}_1,...,\mathbf{r}_N$ respectively. We now assume that (i) the single-agent survival probabilities are identical ie. $s_k=s$ (ii) searchers launch positions are independent and identically distributed, i.e. $\varrho_S(\mathbf{r}_1,\cdots,\mathbf{r}_N)=\prod_{k=1}^{N}\rho_s(\mathbf{r}_k)$. With this we find

$$\bar{S}(T,\{t_i\})=\int d\mathbf{r}_T\varrho_T(\mathbf{r}_T)\left(\prod_{k=1}^{N}\int d\mathbf{r}_k\rho_s(\mathbf{r}_k)s(T-t_k,\mathbf{r}_k,\mathbf{r}_T)\Theta(T-t_k)\right) \tag{3}$$

From here, we want to identify under which conditions equation (3) can be written as

$$\bar{S}(T,\{t_i\})=\prod_{k=1}^{N}\bar{s}(T-t_k)\Theta(T-t_k) \tag{4}$$

This can be achieved in different ways, namely

- for a target fixed at a deterministic position $\mathbf{r}_T^0$, i.e. $\varrho_T(\mathbf{r}_T)=\delta(\mathbf{r}_T-\mathbf{r}_T^0)$. In this case we have $\bar{s}(T-t_k)=\int d\mathbf{r}_k\rho_s(\mathbf{r}_k)s(T-t_k,\mathbf{r}_k,\mathbf{r}_T^0)$
- in a finite search domain $\mathcal{V}$ with translational invariance and periodic boundary conditions such that s depends on $\mathbf{r}_k$ and $\mathbf{r}_T$ through $\mathbf{r}_k-\mathbf{r}_T$, and where agents are launched homogeneously in the domain, i.e. $\rho_s(\mathbf{r}_k)=1/V$ where V is the volume of the search domain. There we obtain

$$\int_{\mathcal{V}}d\mathbf{r}_k\rho_s(\mathbf{r}_k)s(T-t_k,\mathbf{r}_k,\mathbf{r}_T)=\frac{1}{V}\int_{\mathcal{V}+\mathbf{r}_T}d\mathbf{r}_k s(T-t_k,\mathbf{r}_k)=\frac{1}{V}\int_{\mathcal{V}}d\mathbf{r}_k s(T-t_k,\mathbf{r}_k)=\bar{s}(T-t_k) \tag{5}$$

  where the second equality comes from periodicity.

DEFINITION OF THE MEAN SEARCH COST

We define the search cost as the sum of three contributions, i.e. $\bar{K}=\bar{T}+\gamma\bar{\mathcal{T}}+\kappa\bar{N}$ where $\bar{T}$ is the overall mean first-passage time, $\bar{\mathcal{T}}$ is the mean total time spent by searchers in the system and $\bar{N}$ is the mean number of searchers launched. Let us now, explicit these three contributions in terms of the joint survival probability $S(t)$. The mean first-passage time $\bar{T}$ is classically obtained as $\bar{T}=\int_0^\infty\bar{S}(t)dt$. Similarly, as the first-passage time distribution $R(t)$ related to the survival probability $\bar{S}(t)$ via $R(t)=-\bar{S}'(t)$, we have using integration by parts

$$\bar{\mathcal{T}}=\sum_{n=1}^{\infty}\int_0^\infty\frac{d}{dt}\left((t-t_n)\Theta(t-t_n)\right)\bar{S}(t)dt=\sum_{n=1}^{\infty}n\int_{t_n}^{t_{n+1}}\bar{S}(t)dt \tag{6}$$

and finally

$$\bar{N}=\sum_{n=1}^{\infty}\int_0^{\infty}\frac{d\Theta(t-t_n)}{dt}\bar{S}(t)dt=\sum_{n=1}^{\infty}\bar{S}(t_n) \tag{7}$$

TWO SEARCHERS

We first investigate the simple case of two searchers, for which we show that the log-convexity of the SASP $s(t)$ is a sufficient condition for the absence of a local maximum in the quantiles of the search cost, which proves the unicity of a local minimum. For the mean search cost, we show that we also need $-s'$ to be log-convex to prove the unicity of a local minimum.

Quantiles

Let $\mathcal{F}(k)$ be the probability for the search cost K to be less than k. Because of the piecewise affine relation between K and the first-passage time T, we can express $\mathcal{F}$ in terms of the overal survival probility $S(t)$, which is itself a function of the SASP $s(t)$, reading

$$\mathcal{F}(k)=\begin{cases}1 & \text{if } 0\leq k<k_1^-\\ s\left(\frac{k-\kappa}{1+\gamma}\right) & \text{if } k_1^-\leq k<k_1^+\\ s\left(t_2\right) & \text{if } k_1^+\leq k<k_2^-\\ s\left(\frac{k-2\kappa-(1+\gamma)t_2}{1+2\gamma}\right)s\left(\frac{k-2\kappa+\gamma t_2}{1+2\gamma}\right) & \text{if } k>k_2^-\end{cases} \tag{8}$$

with $k_1^-=\kappa$, $k_1^+=k_1^-+(1+\gamma)t_2$ and $k_2^-=k_1^++\kappa$. The z-quantile $k_q(z)$ is defined as the solution of $\mathcal{F}(k_q(z))=z$. We want here to find the optimal introduction time t_2 that minimizes $k_q(z)$ for an arbitrary values of z. Two cases can be identified. If $\mathcal{F}(k_1^+)<z$, then we have $k_1^-<k_q(z)<k_1^+$, i.e.

$$\begin{aligned} s\left(\frac{k_q(z)-k_1^-}{1+\gamma}\right)&=z\\ \Leftrightarrow k_q(z)&=k_1^-+(1+\gamma)s^{-1}(z)\end{aligned} \tag{9}$$

However, if $\mathcal{F}(k_1^+)>z$, then we have $k_q(z)>k_2^-$ such that

$$s\left(\frac{k_q(z)+\gamma t_2-2\kappa}{1+2\gamma}\right)s\left(\frac{k_q(z)-(1+\gamma)t_2-2\kappa}{1+2\gamma}\right)=z \tag{10}$$

To solve this, let us first define $X=\frac{k_q(z)}{1+2\gamma}$, $\tau=\frac{\gamma t_2}{1+2\gamma}$ and $\alpha=1+\frac{1}{\gamma}$, to rewrite this equation as

$$s\left(X+\tau\right)s\left(X-\alpha\tau\right)=z \tag{11}$$

Because X is a simple rescaling of $k_q(z)$, the conditions for the existence of a local maximum obtained on X will transfer directly to $k_q(z)$.

As X is defined as the solution oe equation (11), it intrinsically depends on τ. In particular, we are looking for solutions for which $d_\tau X(\tau)=0$ is verified. Taking the derivative of equation (11) with respect to τ and evaluating it given the condition $X'=0$ yields

$$s'\left(X+\tau\right)s\left(X-\alpha\tau\right)=\alpha s(X+\tau)s'(X-\alpha\tau) \tag{12}$$

Now, we want to know whether this solution is a local maximum or minimum. We therefore take the second derivative of equation (11), reading :

$$\begin{aligned} &X''\left(s'(X+\tau)s(X-\alpha\tau)+s(X+\tau)s'(X-\alpha\tau)\right)\\ &+(1+X')^2s''(X+\tau)s(X-\alpha\tau)+(X'-\alpha)^2s(X+\tau)s''(X-\alpha\tau)\\ &+2(1+X')(X'-\alpha)s'(X+\tau)s'(X-\alpha\tau)=0\end{aligned} \tag{13}$$

Again evaluating it with the conditions $X' = 0$ and condition (12) yields

$$X''(1+\alpha)s(X+\tau)s'(X-\alpha\tau)$$
$$+s''(X+\tau)s(X-\alpha\tau) + \alpha^2 s(X+\tau)s''(X-\alpha\tau)$$
$$-2\alpha s'(X+\tau)s'(X-\alpha\tau) = 0 \tag{14}$$

We therefore need to study the sign of

$$X'' = \frac{s''(X+\tau)s(X-\alpha\tau) + \alpha^2 s(X+\tau)s''(X-\alpha\tau) - 2\alpha s'(X+\tau)s'(X-\alpha\tau)}{-(1+\alpha)s(X+\tau)s'(X-\alpha\tau)} \tag{15}$$

The denominator of the latter formula being always positive, the sign of X'' is the same as the sign of the numerator. This can be obtained by first rewriting the last term as

$$2\alpha s'(X+\tau)s'(X-\alpha\tau) = \alpha s'(X+\tau)s'(X-\alpha\tau) + \alpha s'(X+\tau)s'(X-\alpha\tau) \tag{16}$$
$$= \alpha^2 \frac{s'(X-\alpha\tau)^2 s(X+\tau)}{s(X-\alpha\tau)} + \frac{s'(X+\tau)^2 s(X-\alpha\tau)}{s(X+\tau)} \tag{17}$$

where we have used equation (12). We therefore obtain

$$s''(X+\tau)s(X-\alpha\tau) + \alpha^2 s(X+\tau)s''(X-\alpha\tau) - 2\alpha s'(X+\tau)s'(X-\alpha\tau)$$
$$= \frac{s(X-\alpha\tau)}{s(X+\tau)}\left(s''(X+\tau)s(X+\tau) - s'(X+\tau)^2\right) + \alpha^2 \frac{s(X+\tau)}{s(X-\alpha\tau)}\left(s''(X-\alpha\tau)s(X-\alpha\tau) - s'(X-\alpha\tau)^2\right) \tag{18}$$

If $s(t)$ is logarithmically convex, then $s''(t)s(t) > s'(t)^2$ for all t and therefore $X'' > 0$. If $X(\tau)$ presents an extremum it must therefore be a minimum. However, we can take the derivative of equation (11) and evaluate it for $\tau = 0$ to find

$$X'(0) = \frac{\alpha - 1}{2} = \frac{1}{2\gamma} > 0 \tag{19}$$

By continuity, if $X(\tau)$ presents an extremum, the first one for $\tau > 0$ must be a maximum. This contradicts the previous statement. We therefore conclude that $X(\tau)$ does not present an extremum and is therefore monotonically increasing. We therefore have the following situation. Starting from $t_2 = 0$, $k_q(z)$ increases until it reaches the value k_2^-, which happens at $t_2 = t_2^* = s^{-1}(z)$. Then, it jumps to a value equal to $\kappa + (1+\gamma)s^{-1}(z)$ and then stays constant for $t_2 > t_2^*$. The global minimum is therefore found either at $t_2 = 0$ or at $t_2 = t_2^+$ which we denote as (0) and (+). We therefore have to compare the values of $k_q(z)$ at these two points, reading

$$k_q^{(0)}(z) = 2\kappa + (1+2\gamma)s^{-1}(\sqrt{z}) \tag{20}$$
$$k_q^{(+)}(z) = \kappa + (1+\gamma)s^{-1}(z) \tag{21}$$

To minimize the median search cost, i.e. for $z = 1/2$, one should therefore introduce the second search initially at $t = 0$ if

$$\kappa < \alpha\gamma + \beta \tag{22}$$

where we have defined

$$\alpha = s^{-1}\left(\frac{1}{2}\right) - 2s^{-1}\left(\frac{1}{\sqrt{2}}\right) \tag{23}$$
$$\beta = s^{-1}\left(\frac{1}{2}\right) - s^{-1}\left(\frac{1}{\sqrt{2}}\right) \tag{24}$$

Otherwise, the second searcher should be introduced at $t_2 = (1+2\gamma)s^{-1}(1/2)$. Note that for the median, the long-time behaviour of $s(t)$ does not matter as one only needs to know at which time it reaches 1/2 and 1/4. However, it will matter if one chooses a lower value of z to describe the overall search efficiency.

Mean cost

Let us now investigate the 2-searcher case on the level of the mean search cost and check whether the condition of log-convexity for $s(t)$ is also relevant here. The mean search cost $\bar{K}_2$ of the 2-searcher process reads as follows:

$$\bar{K}_2 = \kappa(1+s(t_2)) + (1+\gamma)\int_0^{t_2} s\,(t)\,dt + (1+2\gamma)\int_0^{\infty} s\,(t)\,s\,(t+t_2)\,dt \tag{25}$$

Upon integration by parts, the first and second derivatives of $\bar{K}$ are given by

$$\bar{K}_2'(t_2) = \kappa s'(t_2) - \gamma s\,(t_2) - (1+2\gamma)\int_0^{\infty} s'\,(t)\,s\,(t+t_2)\,dt \tag{26}$$

$$\bar{K}_2''(t_2) = \kappa s''(t_2) - \gamma s'\,(t_2) - (1+2\gamma)\int_0^{\infty} s'\,(t)\,s'\,(t+t_2)\,dt \tag{27}$$

The extrema of $\bar{K}_2$ are found such that $\bar{K}_2' = 0$, which implies

$$\kappa = \frac{\gamma s(t_2) + (1+2\gamma)\int_0^{\infty} s'\,(t)\,s\,(t+t_2)\,dt}{s'(t_2)} \tag{28}$$

Inserting this into equation (27) for $\bar{K}_2''$ leads to

$$\bar{K}_2''(t_2) = \frac{H_2^{(1)}(t_2) + \gamma H_2^{(2)}(t_2)}{s'(t_2)} \tag{29}$$

where we have defined

$$H_2^{(1)}(t_2) = s''(t_2)\int_0^{\infty} s'\,(t)\,s\,(t+t_2)\,dt - s'(t_2)\int_0^{\infty} s'\,(t)\,s'\,(t+t_2)\,dt \tag{30}$$

$$H_2^{(2)}(t_2) = s(t_2)s''(t_2) + 2s''(t_2)\int_0^{\infty} s'\,(t)\,s\,(t+t_2)\,dt - s'(t_2)^2 - 2s'(t_2)\int_0^{\infty} s'\,(t)\,s'\,(t+t_2)\,dt \tag{31}$$

Let us now show that both these terms are negative. To do this, we first note that these functions are of the form

$$H_2^{(k)}(t_2) = g_2^{(k)}(t_2)^2\,\frac{d}{dt_2}\left(\frac{f_2(t_2)}{g_2^{(k)}(t_2)}\right) \tag{32}$$

where we have defined

$$f_2(t_2) = s'(t_2) \tag{33}$$

$$g_2^{(1)}(t_2) = \int_0^{\infty} s'\,(t)\,s\,(t+t_2)\,dt \tag{34}$$

$$g_2^{(2)}(t_2) = s(t_2) + \int_0^{\infty} s'\,(t)\,s\,(t+t_2)\,dt \tag{35}$$

The sign of $H_2^{(k)}$ is therefore the same as the one of $\frac{d}{dt_2}\left(\frac{f_2}{g_2^{(k)}}\right)$ and the opposite as the one of $\frac{d}{dt_2}\left(\frac{g_2^{(k)}}{f_2}\right)$. Let $h_2^{(k)} = \frac{g_2^{(k)}}{f_2}$ and analyze it for $k=1,2$.

- First, for $H_2^{(1)}$ we have

$$h_2^{(1)}(t_2) = \int_0^{\infty} \frac{s'(t)s(t+t_2)}{s'(t_2)}dt$$

$$\frac{d}{dt_2}h_2^{(1)}(t_2) = \int_0^{\infty} s'(t)\frac{s'(t+t_2)s(t_2) - s''(t_2)s(t+t_2)}{s(t_2)}dt \tag{36}$$

  If s is log-convex, then for any $x,y\in\mathbb{R}$ with $x<y$ it holds $s''(x)s(y) > s'(x)s'(y)$ such that we have $\frac{d}{dt_2}h_2^{(1)}(t_2) > 0$ and hence $H_2^{(1)}(t_2) < 0$.

- For $H_2^{(2)}$, we have

$$\begin{aligned}
h_2^{(2)}(t_2) =&\frac{s(t_2)}{s'(t_2)} + 2\int_0^\infty \frac{s'\left(t\right)s\left(t+t_2\right)}{s'(t_2)}dt \\
=&\frac{2}{s'(t_2)s(t_2)}\frac{s(t_2)^2}{2} + 2\int_0^\infty \frac{s'\left(t\right)s\left(t+t_2\right)}{s'(t_2)}dt \\
=&-\frac{2}{s'(t_2)s(t_2)}\int_0^\infty s'(t+t_2)s(t+t_2)dt + 2\int_0^\infty \frac{s'\left(t\right)s\left(t+t_2\right)}{s'(t_2)}dt \\
=&2\int_0^\infty \frac{\left[s'\left(t\right)s(t_2) - s'(t+t_2)\right]s\left(t+t_2\right)}{s'(t_2)s(t_2)}dt \\
=&-2\int_0^\infty \frac{\left[s\left(t\right)s(t_2) - s(t+t_2)\right]s'\left(t+t_2\right)}{s'(t_2)s(t_2)}dt \\
=&2\int_0^\infty \frac{s(t+t_2) - s\left(t\right)s(t_2)}{s(t_2)}\frac{s'\left(t+t_2\right)}{s'(t_2)}dt \\
=&2\int_0^\infty u_t(t_2)v_t(t_2)
\end{aligned} \tag{37}$$

where we have defined $u_a(x) = \frac{s(x+a)-s(x)s(a)}{s(x)}$ and $v_a(x) = \frac{s'(x+a)}{s'(x)}$. Now, note that for a fixed value of a we have $u_a'(x) = \frac{s'(x+a)s(x)-s(x+a)s'(x)}{s(x)^2}$ and $v_a'(x) = \frac{s''(x+a)s'(x)-s'(x+a)s''(x)}{s'(x)^2}$. Assuming log-convexity for both s and $-s'$, we obtain $u_a'(x) > 0$ and $v_a'(x) > 0$. Since both u_a and v_a are positive, the integrand in h is the product of two positive increasing functions of t_2, which makes h also a positive, increasing function of t_2. This proves that $\frac{d}{dt_2}h_2^{(2)}(t_2) > 0$ such that $H_2^{(2)}(t_2) < 0$.

We therefore obtain $H_2^{(1)} + \gamma H_2^{(2)} < 0$ and thus $\bar{K}_2'' > 0$ for any extremum of $\bar{K}_2$. Any extremum is therefore a local minimum. Now, suppose there are two local minima. By continuity and differentiability, there must also be a local maximum between both of them, which is impossible. We therefore conclude that $\bar{K}_2$ has at most one local minimum and no local maximum.

CONDITION FOR SIMULTANEOUS INTRODUCTION

Here, we want to find under which condition should multiple searchers be launched simultaneously into the system, in the case where $\bar{K}$ does not present any local maximum. To do this, we are using the notations introduced in the previous paragraph on gradient computation.

First, as mentioned in the main text, the minimum of $\bar{K}$ is located at a point $\mathbf{t}_N^*$ where either $t_i^* > t_{i-1}^*$ and $\nabla_i \bar{K}_N(\mathbf{t_N^*}) = 0$, or $t_i^* = t_{i-1}^*$ and $\nabla_i \bar{K}_N(\mathbf{t_N^*}) \geq 0$. Let us now proceed by contradiction. Assume that $\bar{K}_N$ is minimal at a point $\mathbf{t}_N^*$ where, for a certain $k > 2$, we have $t_k^* = t_{k-1}^*$ and $t_{k-1}^* > t_{k-2}^*$. Following our previous observation, this implies that $\nabla_k \bar{K}_N > 0$ and $\nabla_{k-1}\bar{K}_N = 0$. Let us now compute the difference between $\nabla_{k-1}\bar{K}$ and $\nabla_k \bar{K}$ for $t_{k-1} = t_k$. Following the calculation from the previous section, we have

$$\begin{aligned}
\nabla_{k-1}\bar{K} - \nabla_k \bar{K} =& -\gamma(s_{k-1} - s_k) - \sum_{n=k-1}^{\infty}(1+n\gamma)I_{n,k-1} + \sum_{n=k}^{\infty}(1+n\gamma)I_{nk}dt \\
&+ \kappa\left(\sum_{n=1}^{k-2} r_{k-1,n} - \sum_{n=k}^{\infty} r_{n,k-1} - \sum_{n=1}^{k-1} r_{kn} + \sum_{n=k+1}^{\infty} r_{nk}\right)
\end{aligned} \tag{38}$$

For $t_{k-1} = t_k$, we have $s_{k-1} = s_k$ such that most of the terms in all the sums compensate each other and we obtain

$$\left(\nabla_{k-1} - \nabla_k\right)\bar{K} = -2\kappa s'(0)S_k(t_k) \tag{39}$$

Note that this result also holds for a finite value of N as the right-hand side only depends on t_k, such that imposing $t_N \to \infty$ for $N > k$, as one should to deal with $\bar{K}_N$, does not impact it. Because $s'(0) < 0$, we therefore have $\nabla_{k-1}\bar{K}_N > \nabla_k \bar{K}_N$. This is valid at any point $\mathbf{t}_N$, provided that $t_k = t_{k-1}$, and in particular at $\mathbf{t}_N^*$, where we have

$\nabla_k \bar{K}_N > 0$. We therefore have $\nabla_{k-1}\bar{K} > \nabla_k \bar{K} > 0$, which is in contradiction with the original assumption. We therefore conclude that if $t^*_k = t^*_{k-1}$ then for all $j < k$ we must have $t^*_k = t^*_j = 0$: in the optimal strategy, there can not be searchers launched simultaneously later than at the start of the process.

Let us now calculate the optimal number of searchers N_{sim} to be launched at $t = 0$. From our previous argument, we know that at the optimal point, we have $\nabla_k \bar{K}_N > 0$ for $k \leq N_{sim}$ and $\nabla_k \bar{K}_N = 0$ for $k > N_{sim}$. Let us now compute $\nabla_k \bar{K}$ for $t_2 = \cdots = t_k = 0$ for an arbitrary value of k, assuming that $\nabla_p \bar{K} = 0$ for $p > k$. For any $i > 0$ we have

$$\nabla_i \bar{K} = \gamma s_i + \sum_{n=i}^{\infty}(1+n\gamma)I_{ni} - \kappa\left(\sum_{n=1}^{i-1} r_{in} - \sum_{n=i+1}^{\infty} r_{ni}\right) = 0 \tag{40}$$

The condition $\nabla_p \bar{K} = 0$ for $p > k$ therefore reads

$$\gamma s_p + \sum_{n=p}^{\infty}(1+n\gamma)I_{np} - \kappa\left(\sum_{n=1}^{p-1} r_{pn} - \sum_{n=p+1}^{\infty} r_{np}\right) = 0 \tag{41}$$

Then, to evaluate $\nabla_k \bar{K}$ with $t_2 = \cdots = t_k = 0$, we note that some terms can be simplified, namely

$$s_k = 1 \tag{42}$$

$$I_{nk} = \int_{t_n}^{t_{n+1}} s'(t)s(t)^{k-1} \prod_{p=k+1}^{n} s(t-t_p)dt \text{ for } n > k \tag{43}$$

$$\sum_{n=1}^{k-1} r_{kn} = (k-1)s'(0) \text{ for } n > k \tag{44}$$

The integral term I_{nk} can be transformed using integration by parts, yielding

$$I_{nk} = \left[\frac{s(t)^k}{k}\prod_{p=k+1}^{n} s(t-t_p)\right]_{t_n}^{t_{n+1}} - \frac{1}{k}\sum_{p=k+1}^{n} I_{np} = \frac{1}{k}\left[s_{n+1} - s_n\right] - \frac{1}{k}\sum_{p=k+1}^{n} I_{np} \tag{45}$$

Now, summing over all values of $n \geq k$ yields

$$\sum_{n=k}^{\infty}(1+n\gamma)I_{nk} = -\frac{1}{k} + \frac{\gamma}{k}\sum_{n=k}^{\infty}\gamma n\left[s_{n+1} - s_n\right] - \frac{1}{k}\sum_{p=k+1}^{\infty}\sum_{n=p}^{\infty}(1+n\gamma)I_{np} \tag{46}$$

where we have used the fact that $s_k = 1$. Now, we note that the summand in the last term is exactly the second term in the gradient of $\nabla_p \bar{K}$ in equation (41). Summing equation (41) over $p > k$ yields

$$\sum_{p=k+1}^{\infty}\sum_{n=p}^{\infty}(1+n\gamma)I_{np} = -\sum_{p=k+1}^{\infty}\gamma s_p + \kappa\sum_{p=k+1}^{\infty}\left(\sum_{n=1}^{p-1} r_{pn} - \sum_{n=p+1}^{\infty} r_{np}\right) \tag{47}$$

Let us now rewrite the terms proportional to κ. First, let us note $r_{pn} = \zeta(t_p - t_n)S_p(t_p)$. Swapping the order of summation leads to

$$\begin{aligned}
\sum_{p=k+1}^{\infty}\left(\sum_{n=1}^{p-1} r_{pn} - \sum_{n=p+1}^{\infty} r_{np}\right) &= \sum_{p=k+1}^{\infty}\left(\sum_{n=1}^{k} r_{pn} + \sum_{n=k+1}^{p-1} r_{pn} - \sum_{n=p+1}^{\infty} r_{np}\right) \\
&= \sum_{p=k+1}^{\infty}\sum_{n=1}^{k} r_{pn} + \sum_{n=k+1}^{\infty}\sum_{p=n+1}^{\infty} r_{pn} - \sum_{p=k+1}^{\infty}\sum_{n=p+1}^{\infty} r_{np} \\
&= \sum_{p=k+1}^{\infty}\sum_{n=1}^{k} r_{pn} \\
&= k\sum_{p=k+1}^{\infty} r_{pk}
\end{aligned} \tag{48}$$

where we have used the fact that $r_{pn} = r_{pk}$ for $n \leq k$. We therefore obtain

$$\sum_{p=k+1}^{\infty} \sum_{n=p}^{\infty} (1 + n\gamma) I_{np} = - \sum_{p=k+1}^{\infty} \gamma s_p + k\kappa \sum_{p=k+1}^{\infty} r_{pk} \tag{49}$$

Inserting this into equation (46) leads to

$$\sum_{n=k}^{\infty} (1 + n\gamma) I_{nk} = - \frac{1}{k} + \frac{\gamma}{k} \sum_{n=k}^{\infty} n \left[s_{n+1} - s_n\right] + \frac{\gamma}{k} \sum_{p=k+1}^{\infty} s_p - \kappa \sum_{p=k+1}^{\infty} r_{pk} \tag{50}$$

We note that the last term of the latter equation compensates exactly the last term of $\nabla_k \bar{K}$ in equation (**??**). Bringing everything together yields

$$\nabla_k \bar{K} = - \gamma + \frac{1}{k} - \frac{\gamma}{k} \sum_{n=k}^{\infty} n \left[s_{n+1} - s_n\right] - \frac{\gamma}{k} \sum_{n=k+1}^{\infty} s_n + \kappa (k-1) s'(0) \tag{51}$$

Now, we rewrite

$$\sum_{n=k}^{\infty} n \left[s_{n+1} - s_n\right] = -k - \sum_{n=k+1}^{\infty} s_n \tag{52}$$

to finally obtain

$$\nabla_k \bar{K} = \frac{1}{k} + \kappa (k-1) s'(0) \tag{53}$$

N_{sim} is therefore found as the largest value of k which is such that this quantity is positive.

TEST CASES

We show in figure 1 the six different single-agent survival probabilities that were tested numerically in the paper, i.e.

$$s_{sharp,alg}(t) = \frac{2}{\pi} \arcsin\left(\frac{1}{\left(1 + \frac{\lambda t}{\theta}\right)^{\theta}}\right) \tag{54}$$

$$s_{mild,alg}(t) = \frac{1}{\left(1 + \frac{\lambda t}{\theta}\right)^{\theta}} \tag{55}$$

$$s_{flat,alg}(t) = \sin\left(\frac{\pi}{2\left(1 + \frac{\lambda t}{\theta}\right)^{\theta}}\right) \tag{56}$$

$$s_{sharp,exp}(t) = \frac{2}{\pi} \arcsin\left(e^{-\lambda t}\right) \tag{57}$$

$$s_{mild,exp}(t) = e^{-\lambda t} \tag{58}$$

$$s_{flat,exp}(t) = \sin\left(\frac{\pi}{2} e^{-\lambda t}\right) \tag{59}$$

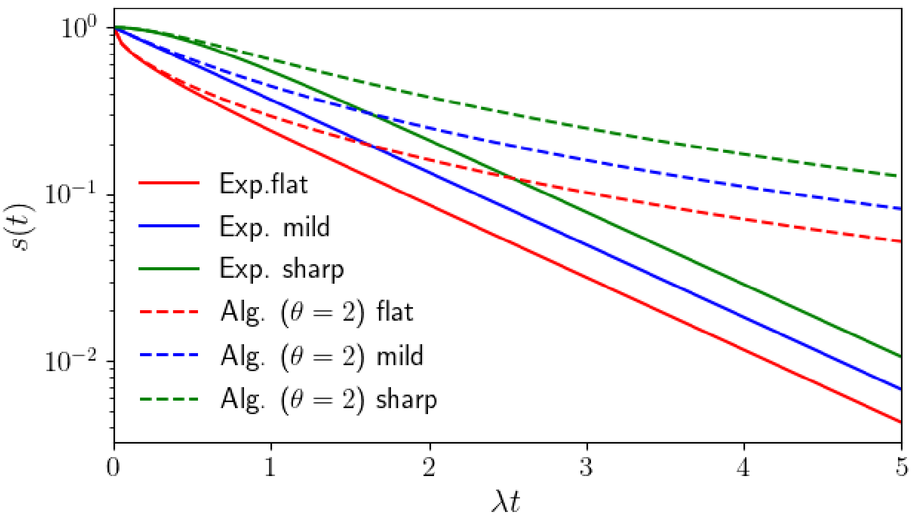

FIG. 1. Test cases for the single-agent survival probabilities.

SEARCH COST IN THE EXPONENTIAL CASE

Let $s(t) = e^{-\lambda t}$. The collective mean first-passage time $\bar{T}$ can be calculated analytically upon integration and rearrangements of sums to find

$$\bar{T} = \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \prod_{k=1}^{n} e^{-\lambda(t-t_k)} dt = \lambda^{-1} \left[1 - \sum_{n=2}^{\infty} \frac{e^{-\lambda \sum_{k=1}^{n-1} t_n - t_k}}{n(n-1)} \right] \tag{60}$$

$$\bar{\mathcal{T}} = \sum_{n=1}^{\infty} n \int_{t_n}^{t_{n+1}} \prod_{k=1}^{n} e^{-\lambda(t-t_k)} dt = \lambda^{-1} \tag{61}$$

$$\bar{N} = \sum_{n=1}^{\infty} \prod_{k=1}^{n} e^{-\lambda(t-t_k)} = 1 + \sum_{n=2}^{\infty} e^{-\lambda \sum_{k=1}^{n} (t_n - t_k)} \tag{62}$$

Now, note that $\sum_{k=1}^{n} (t_n - t_k) = \sum_{l=1}^{n-1} l \Delta_l$, where we have defined $\Delta_k = t_{k+1} - t_k$. The total cost is thus given by

$$\bar{K} = \lambda^{-1}(1+\gamma) + \kappa + \sum_{n=2}^{\infty} e^{-\lambda \sum_{l=1}^{n-1} l \Delta_l} \left(\kappa - \frac{\lambda^{-1}}{n(n-1)} \right) \tag{63}$$

The partial derivative with respect to Δ_k reads then

$$\frac{\partial \bar{K}}{\partial \Delta_k} = \lambda k \sum_{n=k+1}^{\infty} e^{-\lambda \sum_{l=1}^{n-1} l \Delta_l} \left(\frac{\lambda^{-1}}{n(n-1)} - \kappa \right) \tag{64}$$

Here, we note that the terms for which $\frac{\lambda^{-1}}{n(n-1)} > \kappa$ will yield positive contributions to the gradient. Let n^* be the lowest value of n such that this quantity is negative. Then, for any $k \geq n^*$ the gradient $\frac{\partial \bar{K}}{\partial \Delta_k}$ will be negative such that the optimal strategy should be such that $\Delta_k \to \infty$. In this case, the contributions of all terms with $n \geq n^*$ will vanish because of the exponential factor. Therefore, for all $k < n^*$ the gradient will be positive such that the cost will be minimized for $\Delta_k \to 0$. The optimal strategy is therefore such that $N_{sim} = \left\lfloor \frac{1}{2} \left(1 + \sqrt{1 + \frac{4}{\kappa \lambda}} \right) \right\rfloor$ searchers should be launched at $t = 0$ and none later. In this case, the mean search cost is found as

$$\bar{K}_{opt} = \frac{1}{N_{sim}} + \gamma + \kappa \lambda N_{sim} \tag{65}$$

Neglecting the floor part in N_{sim} we finally obtain

$$\bar{K}_{opt} = \gamma + \sqrt{\kappa\lambda}\left(\frac{2}{\sqrt{\kappa\lambda}+\sqrt{\kappa\lambda+4}} + \frac{\sqrt{\kappa\lambda}+\sqrt{\kappa\lambda+4}}{2}\right) \tag{66}$$

As $\kappa \to 0$, the terms between brackets will tend to 1 such that the overall cost will grow as $\sqrt{\kappa\lambda}$. However, as $\kappa\lambda \gg 1$, the brackets will be dominated by the second term which will tend to $\sqrt{\kappa\lambda}$, making the cost grow linearly with $\kappa\lambda$.

COST FOR RESETTING

Let us recall the result of Chechkin et al. [16]. Consider a stochastic search process for which the first-passage time distribution is notes p_T. Now, assume that the search is re-initialized after a time τ drawn from a probability distribution p_τ. In this case, the overall first-passage time probability distribution $\mathcal{P}_T$ is given by

$$\mathcal{P}_T(T) = \sum_{n=0}^{\infty} \pi_n \int_0^T dt \Pi_n(t) P_T(T-t) \tag{67}$$

where π_n is the probability have exactly n resets, P_τ is the probability distribution of the time between two resets, P_T is the first-passage time distribution from the last reset and $\Pi_n(t) = \int_0^t du \Pi_{n-1}(u) P_\tau(t-u)$ with $\Pi_0(t) = 2\delta(t)$. In addition, we have

Now, let us compute π_n, P_τ and P_T. First, we have

$$\pi_n = \pi_0(1-\pi_0)^n \tag{68}$$

$$\pi_0 = \int_0^\infty d\tau p_\tau(\tau) \int_0^\tau dT p_T(T) \tag{69}$$

$$P_\tau(t) = \frac{p_\tau(t)\int_t^\infty dT p_T(T)}{1-\pi_0} \tag{70}$$

$$P_T(t) = \frac{p_T(t)\int_t^\infty p_\tau(\tau) d\tau}{\pi_0} \tag{71}$$

Going in Laplace, equation 67 can be simplified as it ends up being a geometric series. The mean first-passage time is therefore found using the generating function of P_T to find

$$\bar{T} = \frac{\int_0^\infty d\tau p_\tau(\tau) G(\tau)}{\int_0^\infty d\tau p_\tau(\tau) F_T(\tau)} \tag{72}$$

where we have defined $F_T(t) = \int_0^t ds p_T(s)$ and $G(t) = t - \int_0^t dT F_T(T)$. By virtue of the mean value theorem, there exists $t^* \in \mathbb{R}^+$ such that

$$\bar{T} = \frac{G(t^*)}{F_T(t^*)} \frac{\int_0^\infty d\tau p_\tau(\tau) F_T(\tau)}{\int_0^\infty d\tau p_\tau(\tau) F_T(\tau)} = \frac{G(t^*)}{F_T(t^*)} \tag{73}$$

such that

$$\bar{T} \geq \min_t \frac{G(t)}{F_T(t)} \tag{74}$$

This lower bound is in particular realized for $p_\tau(\tau) = \delta(\tau - t^*)$ and with $t^* = \text{argmin}_t \frac{G(t)}{F_T(t)}$.

Let us know define the overall search cost consistently with the one defined for launching new searchers. A first contribution $(1+\gamma_r)\bar{T}$ quantifies the overall cost due to the presence of the target and the cost for sustaining the searcher. A second contribution $\bar{k}_r$ is defined as a mean resetting cost $\kappa_r \bar{N}$, i.e. the cost for a resetting event multiplied by the mean number of resetting events.

Fixed cost Let us first associate a fixed cost κ_r to each resetting event and to launching the searcher in the first place. The mean cost for resetting is therefore given by

$$\bar{k}_r = \kappa_r \left(1 + \sum_{n=0} n\pi_n\right) = \kappa \left(1 + \pi_0 \sum_{n=0} n(1-\pi_0)^n\right) = \frac{\kappa}{\pi_0} \tag{75}$$

The total mean search cost is defined as $\bar{K} = (1+\gamma_r)\bar{T} + \bar{k}_r$, i.e.

$$\bar{K} = \frac{(1+\gamma_r)\int_0^\infty d\tau p_\tau(\tau)G(\tau) + \kappa_r}{\int_0^\infty d\tau p_\tau(\tau)F_T(\tau)} \tag{76}$$

Now, note that one can also write this as

$$\bar{K} = \frac{\int_0^\infty d\tau p_\tau(\tau)\left[(1+\gamma_r)G(\tau) + \kappa_r\right]}{\int_0^\infty d\tau p_\tau(\tau)F_T(\tau)} \tag{77}$$

such that the argument involving the mean value theorem still holds, i.e. there exists a time Δ such that

$$\bar{K} = \frac{(1+\gamma_r)G(\Delta) + \kappa_r}{F_T(\Delta)} \tag{78}$$

This is realized in particular for $p_\tau(\tau) = \delta(\tau - \Delta)$.

Linear cost Let us now define the resetting cost such that is proportional to the distance that the searcher has reached from the origin upon resetting. For 1-dimensional processes, let $\rho_b(x,t)$ be the probability distribution for the searcher to be at position x at position t with the absorbing boundary condition $\rho_b(x_T,t) = 0$. Let us note $\rho(x,t) = c(x,t)/\int_{-\infty}^{x_T} dx c(x,t)$ where $c(x,t)$ is the "concentration" in an experiment with absorption, i.e. $\int_{-\infty}^{x_T} dx c(x,t) = s(t) = 1 - F_T(t)$. Summing over all resetting events, the total mean resetting cost

$$\bar{k}_r = \frac{\int_0^\infty d\tau P_\tau(\tau)\int_{-\infty}^{x_T} dx x \frac{c(x,\tau)}{1-F_T(\tau)}}{\pi_0} = \frac{\int_0^\infty d\tau p_\tau(\tau)\int_{-\infty}^{x_T} dx x c(x,\tau)}{\pi_0(1-\pi_0)} \tag{79}$$

Again, in virtue of the mean value theorem, there exists a time Δ such that

$$\bar{K} = \frac{(1+\gamma_r)G(\Delta) + \kappa_r(\Delta)}{F_T(\Delta)} \tag{80}$$

with

$$\kappa_r(\Delta) = \frac{f\int_{-\infty}^{x_T} dx|x|c(x,\Delta)}{1 - \int_0^\infty d\tau p_\tau(\tau)\int_0^\tau dT p_T(T)} \tag{81}$$

For $p_\tau(\tau) = \delta(\tau - \Delta)$ we obtain $\bar{k}_r(\Delta) = f\frac{\int_{-\infty}^{x_T} dx|x|c(x,\Delta)}{F_T(\Delta)}$. note that this results can easily be generalized in higher dimensions. In the case of the 1-dimensional diffusion problem, we have $F_T(\Delta) = 1 - \mathrm{erf}\left(\frac{x_T}{2\sqrt{D\Delta}}\right)$ and $c(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left(e^{-\frac{x^2}{4Dt}} - e^{-\frac{(x-2x_T)^2}{4Dt}}\right)$ [49]. Defining $l_r = \sqrt{D\Delta}$ and $\eta = x_T/l_r$ we obtain

$$\kappa_r = f l_r\left[\frac{2}{\sqrt{\pi}}\left(1 - e^{-\eta^2}\right) + \eta\left(1 - 2\mathrm{erf}(\eta) + \mathrm{erf}\left(\frac{\eta}{2}\right)\right)\right] \overset{\Delta\to\infty}{\to} f x_T \tag{82}$$