Should you hire new searchers?
Optimal number of agents in a collective search, and when to launch them.

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Search processes often involve multiple agents that collectively look for a randomly located target. While increasing the number of agents usually decreases the time at which the first agent finds the target, it also requires resources to create and sustain more agents. In this letter, we consider a collective search cost that not only accounts for the search time but also for the cost associated to the creation and the maintenance of an agent. We first present a general formalism for independent agents in terms of the survival probability of the target for a single-agent search $s(t)$, where we allow agents to be introduced in the system one after the other. From this, we first derive analytically the optimal number of searchers to launch initially in the system. Then, we identify the optimal strategies for exponential and algebraic single-agent survival probabilities by pointing out the ideal times at which new searchers should be launched in the system. Our results show that all searchers should be launched simultaneously in the exponential case, while some should be launched at later times in the algebraic case. Finally, we compare these results with numerical simulations of a strongly interacting collective search, the true self-avoiding walk, and show how the optimal strategy differ from the non-interacting case.

Search processes are encountered in a very wide variety of systems as this term encompasses any phenomenon where one or multiple agents are looking for a target. From immune processes and the search for pathogens [1], to animal foraging and hunting and the search for food [2, 3], or to human search processes such as police investigations, most of these tasks need to be optimized in a certain way. Most frequently, one requires the typical time to find the target to be minimized with respect to the parameters governing the search process and that can be tuned by the searchers. Over the past recent years, a considerable amount of work has been dedicated to identifying efficient and optimal search strategies in various contexts. For individual searchers, it has for instance been shown that intermittent search strategies as well as Lévy walks are particularly efficient and are frequently observed in nature [4–9]. More recently, many studies have investigated the impact of stochastic resetting on search efficiency and have shown that the resetting rate can also be optimized to improve the search efficiency [10–14]. Finally, non-markovian searches have attracted some attention as well, where the influence of memory on the search properties has been studied [15–19].

Beyond the case of individual searches, collective processes have also been extensively studied over the past decade in various contexts: for hunting and foraging or for the global immune response, understanding how the first-passage time properties vary with the number of searchers is important, not only for non-interacting agents [20], but also for systems where searchers can interact and communicate [21–25]. To this day, the number of searchers has however very rarely been considered as a variable to be optimized. While it is clear that the large majority of processes will lead to a monotonically decreasing mean search time as a function of the number of searchers, it must be emphasized that adding more searchers to a process may have a cost. For a human search problem where one needs to pay agents and material resources to perform the search, one can very easily understand that it is not optimal to hire as much agents as possibly available as it would have a huge financial cost. Similarly, in an immune response process, hiring more cells to find a pathogen requires a substantial amount of metabolic energy. This is precisely the motivation for the question that we are raising in this letter: given the cost associated to the hiring and maintenance of an agent in a collective search process, what is the optimal number of them and when should one launch them in to the system?

To answer this, we first formalize the question for non-interacting searchers in terms of the survival probability of the target in a single-agent search. We allow the searchers to be introduced in the system at different times and we define a mean search cost which consists in one contribution from the overall search time, one from the total time spent by searchers in the system and one from the creation/hiring of a searcher. We first derive general results, in particular the optimal number of searchers to introduce simultaneously at the start of the search. We then consider the main two classes of single-agent target survival probabilities, namely exponential and algebraic, for which we optimize the introduction times. Finally, we consider a process with trail-leaving interacting searchers, and we compare its optimal number of searchers to the non-interacting case.

Formalism and definition of search cost. We consider a search process of N independent searchers for a single target, all characterized by a common single-agent survival probability (SASP) of the target, denoted as $s(t)$. We consider in this paper only cases where $\lim_{t\to\infty} s(t) = 0$, i.e. the target will eventually be found with probability 1. At time t_i, the i-th searcher is introduced into the system and performs an independent search without interacting with the searchers introduced previously. The entire process is stopped whenever one of the searchers finds the target. The question that we are raising here is: what are the optimal introduction times t_i that make the search most efficient? In order to respond, we need to be more specific on the meaning of search efficiency. We therefore define the search cost $\mathcal{K}$ as

$$\mathcal{K} = J_T \bar{T} + J_N \bar{\mathcal{T}} + K_N \bar{N} \tag{1}$$

where $\bar{T}$ is the mean search time, $\bar{\mathcal{T}}$ is the mean sum of times spent by all searchers in the system until the target is found, and $\bar{N}$ is the mean number of searchers introduced in the system until the target is found. J_T, J_N an K_N are parameters that weight each contribution to the search cost. They can be interpreted as follows:

- J_T is the *target cost rate*: it quantifies a cost associated to the only presence of the target and can be thought of as a rate of damage due to the presence of the target.
- J_N is the *searcher maintenance rate*: it quantifies the amount of resources required to sustain one searcher per unit time.
- K_N is the *searcher hiring cost*: it quantifies the amount of resources required to introduce a searcher into the system.

The optimal search strategy is then defined by the set of introduction times t_i that minimize $\mathcal{K}$. For compactness, we introduce the normalized parameters $\gamma = J_N/J_T$ and $\kappa = K_N/J_T$ and set $J_T = 1$ as our cost rate unit for the rest of the paper.

To calculate $\bar{T}$, $\bar{\mathcal{T}}$ and $\bar{N}$, we first need to formalize the target survival probability, i.e. the probability that neither of the searchers has found the target yet after a certain time t. For $t_n \leq t < t_{n+1}$, there are n searchers in the system, and we have

$$S(t) = S_n(t) \equiv \prod_{k=1}^{n} s(t - t_k) \tag{2}$$

Note that we choose by convention $t_1 = 0^+$ as the time origin. Following the calculation reported in the Supplemental material, we rewrite $\mathcal{K}$ as

$$\mathcal{K} = \sum_{n=1}^{\infty} \left[(1 + n\gamma) \int_{t_n}^{t_{n+1}} S_n(t) dt + \kappa S_n(t_n) \right] \tag{3}$$

Note that this framework can be used to model a finite number of searchers N simply by imposing $t_p \to \infty$ for $p > N$. In this case, we note the search cost $\mathcal{K}_N$.

It should be emphasized that both S and s share the same level of averaging with respect to the initial positions of the target and of the searchers. If s depends on these positions, S does as well and corresponds to the survival probability when all searchers are initialized at the same position and look for the same target.

Condition for simultaneous introduction. Consider a process with a total reservoir of N searchers. We first want to know whether it is favorable to introduce searchers simultaneously or if one should launch them one after the other into the system.

Let us first assume that $\mathcal{K}_N$ is a convex function of $\mathbf{t}_N = (t_2, \cdots, t_N)$, such that there exists at most one local (and hence global) minimum of the cost function. This assumption is not necessarily true for any SASP $s(t)$, but most standard cases satisfy this hypothesis. Also note that the domain over which the function $\mathcal{K}_N$ is defined is such that $t_i \geq t_{i-1}$ for any $i \geq 2$. Because of convexity, the minimum of $\mathcal{K}$ is therefore located at a point $\mathbf{t}_N^*$ where either $t_i^* > t_{i-1}^*$ and $\nabla_i \mathcal{K}_N(\mathbf{t_N^*}) = 0$, or $t_i^* = t_{i-1}^*$ and $\nabla_i \mathcal{K}_N(\mathbf{t_N^*}) \geq 0$ [26].

Starting from this observation, we show in the Supplemental material that the optimal strategy is such that no searchers are launched simultaneously later than at the start of the process. The only agents introduced together are launched at $t = 0$ and their number N_{sim} is the largest integer k that verifies $\nabla_k \mathcal{K}_N > 0$ at the optimal point, where $\nabla_p \mathcal{K}_N = 0$ for any $p > N_{sim}$. This statement is valid for any value of N and therefore holds as $N \to \infty$. Moreover, we show that for any $k \geq 2$, $\nabla_k \mathcal{K}$ evaluated at $t_k = t_{k-1} = ... = 0$ with the constraint that $\nabla_p \mathcal{K} = 0$ for $p > k$ is given by

$$\nabla_k \mathcal{K} = k^{-1} - (k-1)\kappa r(0) \tag{4}$$

where $r(0) = -s'(0)$ is the initial value of the single-agent first-passage time distribution. N_{sim} is therefore found as the largest integer k for which the latter quantity is positive, i.e. $k(k-1)\kappa r(0) < 1$. This leads to

$$N_{sim} = \left\lfloor \frac{1}{2} \left(1 + \sqrt{1 + \frac{4}{\kappa r(0)}} \right) \right\rfloor \tag{5}$$

This result is very general as it does not rely on strong assumptions on $s(t)$ and shows that N_{sim} does not depend on γ at all. No matter how much it costs to sustain a searcher, the number of searchers to be introduced into the system at $t = 0$ will only be governed by κ.

While N_{sim} can be found analytically in general, finding the optimal values of t_i for $i > N_{sim}$ must be treated specifically for a given survival probability

$s(t)$. Two prominent cases for $\lim_{t\to\infty} s(t) = 0$ can be identified [27, 28]: exponentially or algebraically decaying, both of which we treat separately in the following.

Exponential SASP. Let us consider a single-agent survival probability given by $s(t) = e^{-\lambda t}$. We show in the Supplemental material that the derivative of $\mathcal{K}$ with respect to $\Delta_k = t_{k+1} - t_k$ is given by

$$\partial_{\Delta_k}\mathcal{K} = \lambda k \sum_{n=k+1}^{\infty} e^{-\lambda \sum_{l=1}^{n-1} l\Delta_l}\left(\frac{\lambda^{-1}}{n(n-1)} - \kappa\right) \quad (6)$$

Because the function $1/n(n-1)$ is decreasing with n, all terms such that $n(n-1)\lambda\kappa > 0$ are a negative contribution to the derivative. This condition is exactly the same as the one discussed in equations (4) and (5), with $r(0) = \lambda$. Therefore, for $k+1 > N_{sim}$ as defined in equation 5, all contributions are negative, leading to an overall negative exponential decay as a function of Δ_k for the derivative. This implies that for all $k \geq N_{sim}$, Δ_k must be infinite for $\mathcal{K}$ to be minimized, i.e. at most N_{sim} particles should be introduced in the system.

For $k < N_{sim}$, all terms of $\nabla_k\mathcal{K}$ with $n > N_{sim}$ are vanishingly small as all Δ_l with $l > N_{sim}$ in the exponential need to be infinitely large, as shown previously, leading to a zero contribution to the derivative. The sum therefore runs for $k+1 \leq n \leq N_{sim}$, where all terms are positive. The overall minimum of the function will therefore be found for $\Delta_k = 0$. The best strategy is therefore simple: N_{sim} searchers should be introduced in the system simultaneously at $t = 0$ and none after. As observed previously, this number does not depend on γ, and the optimal cost grows as $2\sqrt{\kappa}$ for $\kappa r(0) \ll 1$ and linearly with κ for $\kappa r(0) \gg 1$ (see Supplemental material for the full equation).

Algebraic SASP. Let us now consider single-agent survival probabilities decaying as $t^{-\theta}$ for $t \to \infty$, where $\theta > 0$ is the so-called survival exponent. Examples of such processes include random walks on fractals [29–31], or in infinite space [32]. Here, we can not calculate the search cost $\mathcal{K}$ analytically as for the exponential case, therefore we tackle it using numerical optimization and consider the following SASP: $s(t) = (1+\lambda t)^{-\theta}$.

For arbitrary $\theta > 0$, the cost function $\mathcal{K}$ should in principle be minimized with respect to all introduction times t_i. In practice, we performed a numerical gradient-descent optimization using N of these times, making sure that the algorithm converges to $S(t_N) < 0.001$. We used three different values of θ, namely 1/2, 1 and 3/2.

The optimal strategy defined from the optimal introduction times t_n was found to differ substantially from the exponential case. Here, N_{sim} searchers should first be introduced in the system, whose value matches perfectly the prediction of equation (5) with $r(0) = \theta\lambda$. Then, the next searchers should be introduced at an almost constant time interval Δ^*. In fact, the value of Δ_n very quickly reaches a constant value as n increases. This optimal interval becomes larger with larger κ and θ but decreases with γ, as shown in figure 1. This is also observed by analyzing the effective optimal number of searchers N_{opt} defined as $S(t_{N_{opt}}) > 0.01 > S(t_{N_{opt}+1})$, whose variation is opposite to the one of Δ^* as shown in figure 2.

As algebraic SASPs correspond to searches where the searching time might be very long, our results show that one should better not start all searchers simultaneously. In fact, if all N_{opt} agents were to be launched at $t = 0$, there would be a non-negligible probability that the target would still take a lot of time to be found even with many searchers in the system, resulting in a large overall cost. The searching resources must therefore be

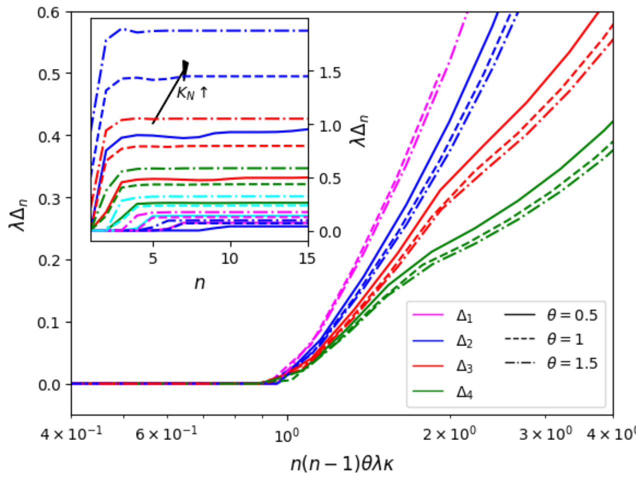

FIG. 1. Optimal values of Δ_n for $n \leq 4$, $\theta = 0.5, 1, 1.5$ and $\gamma = 0.5$ as a function of $n(n-1)\theta\lambda\kappa$. The inset shows the same data as a function of n for various values of κ. The curves in the main panel do not collapse on each other as suggested by the inset because the x-axis depends explicitly on n, which is different for each curve.

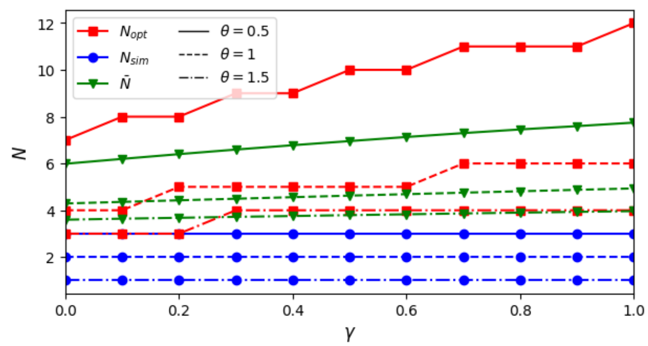

FIG. 2. N_{opt}, N_{sim} and $\bar{N}$ for $\theta = 0.5, 1, 1.5$ and $\kappa\theta\lambda = 0.128$ as a function of γ.

managed carefully and launched into the system progressively only if the first ones struggle to find the target.

Interacting searchers. While the results presented in the previous section hold for non-interacting searchers, one can legitimately wonder to which extent they are still valid for interacting ones. It is unfortunately not possible to derive very general results for arbitrary interacting searchers as the combined target survival probability $S(t)$ would highly depend on the process under study. However, we can still optimize the launching strategy in a particular process where searchers interact strongly and evaluate how the results differ from the non-interacting case.

Here, we investigate a collective search by autochemotactic walkers, as presented in refs. [16, 33]. There, a a field c is defined on a 2D-lattice. At each time step, each walker first adds an amount δc to the field c on the site it occupies. Then, the field diffuses over a certain duration following normal diffusion parametrized by a diffusion coefficient D_c. Finally, each walker jumps to a neighboring site j with a probability proportional to $e^{-\beta c_j}$, where $\beta > 0$ quantifies the chemotactic coupling strength. For $D_c = 0$, the model reduces to the true self-avoiding walk [34] where the system retains an infinite memory as the trace left by walkers along their paths never decays.

We run Monte-Carlo simulations of this model on a square lattice of size 100×100. Initially, a random site of the lattice is identified as a target site. All walkers are also introduced into the system at random lattice sites. We first ran simulations for a single walker, which we always found to yield an exponential SASP. For the N-searcher problem, we first launched N_{sim} walkers in the system at $t = 0$ and then introduced additional ones one by one at a constant time interval Δ until one of the walkers reaches the target. We show in figure 3 an example of the resulting mean cost $\mathcal{K}$ as a function of N_{sim} and Δ for $D_c = 0$, $\beta = 1$, $\gamma = 0.5$ and $\kappa\lambda = 0.01$ where λ^{-1} is the timescale of the corresponding SASP. The minimum cost is here reached for $N_{sim} = 9$ and $\Delta = 4\times10^3$. We observe that the optimal value of Δ is so large that the mean number of searchers $\bar{N}$ is only very slightly larger than N_{sim}, i.e. it is very rare that the first N_{sim} walkers will not find the target before the next searcher is launched. This is consistent with the prediction made in the previous section for non-interacting searches with an exponential SASP, for which $N_{sim} = \bar{N}$. We also remark that strategies with lower values of Δ and N_0 yielding the same value for $\bar{N}$ lead to a substantially larger search cost, indicating that the mean number of searchers is not the only quantity that matters but the details of the launching strategy have a major impact.

We also display in figure 3 the optimal number N_{sim} as we impose $\Delta \to \infty$ and show that it follows the same trend as the prediction of equation (5) as a function of

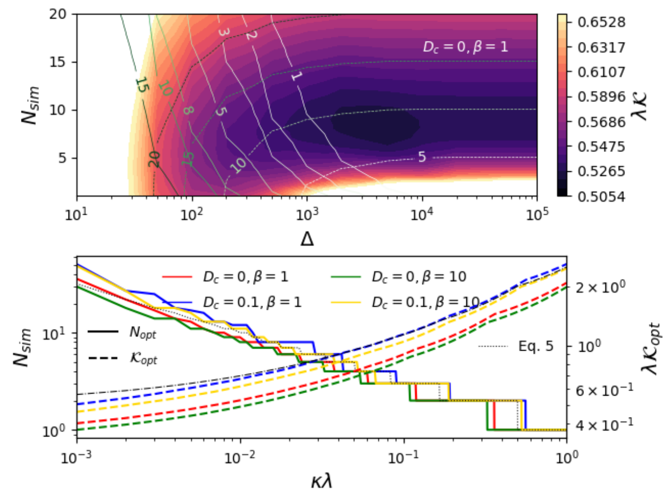

FIG. 3. Upper panel: Mean search cost of the autochemotactic search in the (Δ, N_{sim})-plane, for $D_c = 0, \beta = 1$ and $\gamma = 0.5, \kappa\lambda = 0.01$. The lines show iso-curves of $\bar{N}$ (dotted) and $\bar{N} - N_{sim}$ (solid). Lower panel: Optimal number of walkers N_{sim} for $\Delta \to \infty$ as a function of $\kappa\lambda$ for various values of D_c and β (solid lines) as well as equation (5) (dotted lines). On the second y-axis, we report the associated optimal cost (dashed lines) together with the optimal cost for independent searchers (dash-dotted line).

$\kappa\lambda$. The exact values differ slightly but still remain close, even for the most strongly interacting searchers that we have tested, i.e. for $D_c = 0$ and $\beta = 10$. The value of the optimal search cost $\bar{\mathcal{K}} = \lambda\mathcal{K}$ also follow the same trend but we clearly see that the cost becomes smaller for more strongly interacting searchers. This implies that, although optimal search costs may be reduced thanks to interactions, the optimal launching strategy derived for independent searchers is very likely to be quasi-optimal for interacting searchers.

Discussion. In this letter, we have identified the optimal strategy for when to launch new agents in a search process. First, for non-interacting searchers, if the single-agent survival probability of the target is exponentially decaying, a well-defined number of agents should be launched initially and none later. If it decays algebraically, some agents should also be introduced initially but additional ones should then be launched at an almost constant rate. For interacting searchers, we have shown on one example that the optimal strategy derived for non-interacting ones is very close to the actual optimal one, suggesting that the non-interacting optimal strategies are reliable reference procedures.

As a direction for future works, we make the following points. First, as mentioned originally, our results apply if $s(t)$ and $S(t)$ share the same level of averaging with respect to the original positions of the searchers $\mathbf{r}_s$ and the target $\mathbf{r}_0$. Assume $s(t)$ describes the SASP where $\mathbf{r}_s$ and

$\mathbf{r}_0$ are already averaged, then the definition of $S(t)$ made in equation (2) corresponds to a process where each new agent is initialized randomly following the distribution used in $s(t)$. However, if one considers a process where each new agent is initialized at the same location and one averages over all these possible processes, then our formalism should be adapted. This could for instance be the case for a process where one knows that the target is at a certain distance from the launching center, but not precisely where.

More generally, this observation naturally raises the question of the use of the information available to adapt optimal introduction strategy, as it has for instance been studied for stochastic resetting processes [14, 35]. One could for instance consider a process where one adapts the launching of new agents based on the history of previous walkers, leading to non-markovian decision-making. One can also wonder about the optimal launching locations, in addition to the optimal introduction times, i.e. where should new agents be launched given the location of previous agents. We believe that this letter is a first step towards the formalization of such questions, which will be tackled in future studies.

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Supplement Material

DEFINITION OF SEARCH COST

First, let us express the three contributions of the search cost in terms of $S(t)$. First we have

$$\begin{aligned}\bar{T} &= \int_0^\infty tR(t)dt = -\int_0^\infty tS'(t)dt \\ &= \int_0^\infty S(t)dt \end{aligned} \tag{7}$$

Then, we have

$$\begin{aligned}\bar{\mathcal{T}} &= \sum_{n=1}^{\infty}\int_0^\infty \frac{d}{dt}\left((t-t_n)\Theta(t-t_n)\right)S(t)dt \\ &= \sum_{n=1}^{\infty}\left[\int_0^\infty (t-t_n)\delta(t-t_n)S(t)dt \right. \\ &\qquad \left. + \int_0^\infty \Theta(t-t_n)S(t)dt\right] \\ &= \sum_{n=1}^{\infty}\int_0^\infty \Theta(t-t_n)S(t)dt \\ &= \sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\int_{t_m}^{t_{m+1}} S_m(t)dt \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{m}\int_{t_m}^{t_{m+1}} S_m(t)dt \\ &= \sum_{m=1}^{\infty} m\int_{t_m}^{t_{m+1}} S_m(t)dt \end{aligned} \tag{8}$$

Finally, we have

$$\begin{aligned}\bar{N} &= \int_0^\infty \Theta(t-t_n)R(t)dt \\ &= \sum_{n=1}^{\infty}\int_0^\infty \delta(t-t_n)S(t)dt \\ &= \sum_{n=1}^{\infty} S(t_n) = \sum_{n=1}^{\infty} S_n(t_n) \end{aligned} \tag{9}$$

SEARCH COST IN THE EXPONENTIAL CASE

General case

In the exponential case, we have $s(t) = e^{-\lambda t}$. Let us first compute the collective mean first-passage time $\bar{T}$.

$$\begin{aligned}\bar{T} &= \sum_{n=1}^{\infty}\int_{t_n}^{t_{n+1}}\prod_{k=1}^{n} s(t-t_k)dt \\ &= \sum_{n=1}^{\infty}\int_{t_n}^{t_{n+1}}\prod_{k=1}^{n} e^{-\lambda(t-t_k)}dt \\ &= \sum_{n=1}^{\infty}\int_{t_n}^{t_{n+1}} e^{-\lambda\sum_{k=1}^{n}(t-t_k)}dt \\ &= \sum_{n=1}^{\infty} e^{\lambda\sum_{k=1}^{n}t_k}\int_{t_n}^{t_{n+1}} e^{-\lambda nt}dt \\ &= -\lambda^{-1}\sum_{n=1}^{\infty}\frac{e^{-\lambda\sum_{k=1}^{n}t_{n+1}-t_k}}{n} - \frac{e^{-\lambda\sum_{k=1}^{n}t_n-t_k}}{n} \\ &= -\lambda^{-1}\sum_{n=1}^{\infty}\frac{e^{-\lambda\sum_{k=1}^{n}t_{n+1}-t_k}}{n} - \frac{e^{-\lambda\sum_{k=1}^{n-1}t_n-t_k}}{n} \\ &= -\lambda^{-1}\left[\sum_{n=2}^{\infty}\frac{e^{-\lambda\sum_{k=1}^{n-1}t_n-t_k}}{n-1} - \sum_{n=1}^{\infty}\frac{e^{-\lambda\sum_{k=1}^{n-1}t_n-t_k}}{n}\right] \\ &= \lambda^{-1}\left[1-\sum_{n=2}^{\infty}\frac{e^{-\lambda\sum_{k=1}^{n-1}t_n-t_k}}{n(n-1)}\right] \end{aligned} \tag{10}$$

Similarly, we have

$$\begin{aligned}\bar{\mathcal{T}} &= \sum_{n=1}^{\infty} n\int_{t_n}^{t_{n+1}}\prod_{k=1}^{n} s(t-t_k)dt \\ &= -\lambda^{-1}\sum_{n=1}^{\infty} n\left(\frac{e^{-\lambda\sum_{k=1}^{n}t_{n+1}-t_k}}{n} - \frac{e^{-\lambda\sum_{k=1}^{n}t_n-t_k}}{n}\right) \\ &= -\lambda^{-1}\left[\sum_{n=2}^{\infty} e^{-\lambda\sum_{k=1}^{n-1}t_n-t_k} - \sum_{n=1}^{\infty} e^{-\lambda\sum_{k=1}^{n-1}t_n-t_k}\right] \\ &= \lambda^{-1} \end{aligned} \tag{11}$$

Finally, we compute $\bar{N}$ as

$$\begin{aligned}\bar{N} &= \sum_{n=1}^{\infty}\prod_{k=1}^{n} s(t_n-t_k) \\ &= \sum_{n=1}^{\infty}\prod_{k=1}^{n} e^{-\lambda(t_n-t_k)} \\ &= \sum_{n=1}^{\infty} e^{-\lambda\sum_{k=1}^{n}(t_n-t_k)} \\ &= 1+\sum_{n=2}^{\infty} e^{-\lambda\sum_{k=1}^{n}(t_n-t_k)} \end{aligned} \tag{12}$$

The total cost is thus given by

$$\mathcal{K} = \lambda^{-1}(1+\gamma) + \kappa + \sum_{n=2}^{\infty} e^{-\lambda \sum_{k=1}^{n}(t_n - t_k)} \left(\kappa - \frac{\lambda^{-1}}{n(n-1)}\right) \tag{13}$$

Now, note that

$$\begin{aligned}
\sum_{k=1}^{n}(t_n - t_k) &= n t_n - \sum_{k=1}^{n} t_k \\
&= n\sum_{k=1}^{n-1}\Delta_k - \sum_{k=1}^{n}\sum_{l=1}^{k-1}\Delta_l && (14)\\
&= n\sum_{k=1}^{n-1}\Delta_k - \sum_{l=1}^{n-1}\sum_{k=l+1}^{n}\Delta_l && (15)\\
&= n\sum_{k=1}^{n-1}\Delta_k - \sum_{l=1}^{n-1}(n-l)\Delta_l && (16)\\
&= \sum_{l=1}^{n-1} l\Delta_l && (17)
\end{aligned}$$

where we have defined $\Delta_k = t_{k+1} - t_k$. We thus obtain

$$\mathcal{K} = \lambda^{-1}(1+\gamma) + \kappa + \sum_{n=2}^{\infty} e^{-\lambda \sum_{l=1}^{n-1} l\Delta_l} \left(\kappa - \frac{\lambda^{-1}}{n(n-1)}\right) \tag{18}$$

Now, we take the derivative with respect to Δ_k. In the sum, only terms with $k \le n-1$ will yield a non-zero contribution. We obtain

$$\partial_{\Delta_k}\mathcal{K} = \lambda k \sum_{n=k+1}^{\infty} e^{-\lambda \sum_{l=1}^{n-1} l\Delta_l} \left(\frac{\lambda^{-1}}{n(n-1)} - \kappa\right) \tag{19}$$

Optimal cost

In the optimal case, we proved that there should be N_{sim} searchers launched at $t=0$ and none after. We therefore have

$$\begin{aligned}
\mathcal{K}_{opt} &= \bar{T}_{opt} + \gamma\lambda^{-1} + \kappa N_{sim} \\
&= \lambda^{-1}\left[1 - \sum_{n=2}^{N_{sim}+1}\frac{1}{n(n-1)}\right] + \gamma\lambda^{-1} + \kappa N_{sim} \\
&= \lambda^{-1}\left[1 + \frac{1}{N_{sim}} - 1\right] + \gamma\lambda^{-1} + \kappa N_{sim} \\
\lambda\mathcal{K}_{opt} &= \frac{1}{N_{sim}} + \gamma + \kappa\lambda N_{sim}
\end{aligned} \tag{20}$$

We rewrite $N_{sim} = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\kappa\lambda}}\right)$ as

$$\begin{aligned}
N_{sim} &= \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\kappa\lambda}}\right) \\
&= \frac{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}}{2\sqrt{\kappa\lambda}} \\
\kappa\lambda N_{sim} &= \sqrt{\kappa\lambda}\frac{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}}{2} \\
\mathcal{K}_{opt} &= \gamma + \frac{2\sqrt{\kappa\lambda}}{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}} + \sqrt{\kappa\lambda}\frac{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}}{2} \\
\mathcal{K}_{opt} &= \gamma + \sqrt{\kappa\lambda}\left(\frac{2}{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}} + \frac{\sqrt{\kappa\lambda} + \sqrt{\kappa\lambda + 4}}{2}\right)
\end{aligned} \tag{21}$$

As $\kappa \to 0$, the terms between brackets will tend to 1 such that the overall cost will grow as $\sqrt{\kappa\lambda}$. However, as $\kappa\lambda \gg 1$, the brackets will be dominated by the second term which will tend to $\sqrt{\kappa\lambda}$, making the cost grow linearly with $\kappa\lambda$.

GRADIENT COMPUTATION

We calculate here the gradient of $\mathcal{K}$ with respect to an arbitrary introduction times t_i. First, we have

$$\begin{aligned}
\nabla_p\mathcal{K} &= -\gamma S_{p-1}(t_p) \\
&\quad + \sum_{n=1}^{\infty}\left[(1+n\gamma)\int_{t_n}^{t_{n+1}}\nabla_p S_n(t)dt + \kappa\nabla_p(S_n(t_n))\right]
\end{aligned} \tag{22}$$

The first term comes from evaluating the integrand in $t = t_p$ for $n = p$ and $n+1 = p$. Then we have

$$\begin{aligned}
\nabla_p S_n(t) &= \partial_{t_p}\left[s(t-t_n)s(t-t_{n-1})\cdots s(t-t_2)s(t)\right] \\
&= \begin{cases} -s'(t-t_p)\prod_{\substack{k=1\\k\neq p}}^{n} s(t-t_k) & \text{if } n \ge p \\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{23}$$

Then

$$\begin{aligned}
\nabla_p(S_n(t_n)) &= \partial_{t_p}\left[s(t_n-t_{n-1})\cdots s(t_n-t_2)s(t_n)\right] \\
&= \begin{cases} -s'(t_n-t_p)\prod_{\substack{k=1\\k\neq p}}^{n} s(t_n-t_k) & \text{if } n > p \\ \sum_{k=1}^{p-1} s'(t_p-t_k)\prod_{\substack{l=1\\k\neq l}}^{p-1} s(t_p-t_l) & \text{if } p = n \\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{24}$$

We therefore obtain

$$\begin{aligned}\nabla_p\mathcal{K} = &-\gamma S_p(t_p)\\ &-\sum_{n=p}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} s'(t-t_p)\prod_{\substack{k=1\\k\neq p}}^{n} s(t-t_k)dt\\ &+\kappa\sum_{n=1}^{p-1} s'(t_p-t_n)\prod_{\substack{l=1\\l\neq n}}^{p-1} s(t_p-t_l)\\ &-\kappa\sum_{n=p+1}^{\infty} s'(t_n-t_p)\prod_{\substack{k=1\\k\neq p}}^{n} s(t_n-t_k)\end{aligned} \tag{25}$$

Let $\zeta(t) = \frac{d\ln s(t)}{dt}$. We finally obtain

$$\begin{aligned}\nabla_p\mathcal{K} = &-\gamma S_p(t_p)\\ &-\sum_{n=p}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} \zeta(t-t_p)S_n(t)dt\\ &+\kappa\left(\sum_{n=1}^{p-1}\zeta(t_p-t_n)S_p(t_p)-\sum_{n=p+1}^{\infty}\zeta(t_n-t_p)S_n(t_n)\right)\end{aligned} \tag{26}$$

CONDITION FOR SIMULTANEOUS INTRODUCTION

Here we want to find under which condition should multiple searchers be launched simultaneously into the system. As mentioned in the main text, the minimum of $\mathcal{K}$ is located at a point $\mathbf{t}_N^*$ where either $t_i^* > t_{i-1}^*$ and $\nabla_i\mathcal{K}_N(\mathbf{t}_\mathbf{N}^*) = 0$, or $t_i^* = t_{i-1}^*$ and $\nabla_i\mathcal{K}_N(\mathbf{t}_\mathbf{N}^*) \geq 0$.

Let us now proceed by contradiction. Assume that $\mathcal{K}_N$ is minimal at a point $\mathbf{t}_N^*$ where, for a certain $k > 2$, we have $t_k^* = t_{k-1}^*$ and $t_{k-1}^* > t_{k-2}^*$. Following our previous observation, this implies that $\nabla_k\mathcal{K}_N > 0$ and $\nabla_{k-1}\mathcal{K}_N = 0$. Let us now compute the difference between $\nabla_{k-1}\mathcal{K}$ and $\nabla_k\mathcal{K}$ for $t_{k-1} = t_k$. Following the calculation from the previous section, we have

$$\begin{aligned}\nabla_{k-1}\mathcal{K}-\nabla_k\mathcal{K} = &-\gamma S_{k-1}(t_{k-1})+\gamma S_k(t_k)\\ &-\sum_{n=k-1}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}}\zeta(t-t_{k-1})S_n(t)dt+\sum_{n=k}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}}\zeta(t-t_k)S_n(t)dt\\ &+\kappa\left(\sum_{n=1}^{k-2}\zeta(t_{k-1}-t_n)S_{k-1}(t_{k-1})-\sum_{n=k}^{\infty}\zeta(t_n-t_k)S_n(t_n)\right)\\ &-\kappa\left(\sum_{n=1}^{k-1}\zeta(t_k-t_n)S_k(t_k)-\sum_{n=k+1}^{\infty}\zeta(t_n-t_k)S_k(t_k)\right)\end{aligned} \tag{27}$$

Now, note that for any $n > 0$ it holds $S_{n-1}(t_n) = S_n(t_n)$. Using $t_{k-1} = t_k$, we have $S_{k-1}(t_{k-1}) = S_k(t_k)$. With this, we note that most of the terms in all the sums compensate each other and we obtain

$$\begin{aligned}\left(\nabla_{k-1}-\nabla_k\right)\mathcal{K} = &-(1+(k-1)\gamma)\int_{t_{k-1}}^{t_k}\zeta(t-t_k)S_n(t)dt\\ &-\kappa\zeta(t_k-t_k)S_k(t_k)\\ &-\kappa\zeta(t_k-t_{k-1})S_k(t_k)\\ = &-2\kappa\zeta(0)S_k(t_k)\\ = &-2\kappa s'(0)S_k(t_k)\end{aligned} \tag{28}$$

Note that this result also holds for a finite value of N as the right-hand side only depends on t_k, such that imposing $t_N \to \infty$ for $N > k$, as one should to deal with $\mathcal{K}_N$, does not impact it. Because $s'(0) < 0$, we therefore have $\nabla_{k-1}\mathcal{K}_N > \nabla_k\mathcal{K}_N$. This is valid at any point $\mathbf{t}_N$, provided that $t_k = t_{k-1}$, and in particular at $\mathbf{t}_N^*$, where we have $\nabla_k\mathcal{K}_N > 0$. We therefore have $\nabla_{k-1}\mathcal{K} > \nabla_k\mathcal{K} > 0$, which is in contradiction with the original assumption. We therefore conclude that if $t_k^* = t_{k-1}^*$ then for all $j < k$ we must have $t_k^* = t_j^* = 0$: in the optimal strategy, there can not be searchers launched simultaneously later than at the start of the process.

Now, we want to calculate the optimal number of searchers N_{sim} to be launched at $t = 0$. From our previous argument, we know that at the optimal point, we have $\nabla_k\mathcal{K}_N > 0$ for $k \leq N_{sim}$ and $\nabla_k\mathcal{K}_N = 0$ for $k > N_{sim}$. Let us now compute $\nabla_k\mathcal{K}$ for $t_2 = \cdots = t_k = 0$ for an arbitrary value of k, assuming that $\nabla_p\mathcal{K} = 0$ for

$p > k$. This latter condition reads

$$\gamma S_p(t_p) + \sum_{n=p}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} s'(t-t_p)s(t)^k \prod_{\substack{l=k+1\\ l\neq p}}^{n} s(t-t_l)dt - \kappa\left(\sum_{n=1}^{p-1}\zeta(t_p-t_n)S_p(t_p) - \sum_{n=p+1}^{\infty}\zeta(t_n-t_p)S_n(t_n)\right) = 0 \tag{29}$$

Then, we compute $\nabla_k\mathcal{K}$:

$$\nabla_k\mathcal{K} = -\gamma - \sum_{n=k}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} s'(t)s(t)^{k-1}\prod_{p=k+1}^{n} s(t-t_p)dt + \kappa\left((k-1)s'(0) - \sum_{n=k+1}^{\infty} s'(t_n)s(t_n)^{k-1}\prod_{p=k+1}^{n} s(t_n-t_p)\right) \tag{30}$$

First, we rewrite the second term using integration by parts, reading

$$\begin{aligned}\int_{t_n}^{t_{n+1}} s'(t)s(t)^{k-1}\prod_{p=k+1}^{n} s(t-t_p)dt &= \left[\frac{s(t)^k}{k}\prod_{p=k+1}^{n} s(t-t_p)\right]_{t_n}^{t_{n+1}} \\ &\quad - \frac{1}{k}\sum_{p=k+1}^{n}\int_{t_n}^{t_{n+1}} s(t)^k s'(t-t_p)\prod_{\substack{l=k+1\\ l\neq p}}^{n} s(t-t_l)dt \\ &= \frac{1}{k}\left[S_n(t_{n+1}) - S_n(t_n)\right] \\ &\quad - \frac{1}{k}\sum_{p=k+1}^{n}\int_{t_n}^{t_{n+1}} s(t)^k s'(t-t_p)\prod_{\substack{l=k+1\\ l\neq p}}^{n} s(t-t_l)dt\end{aligned} \tag{31}$$

Now, summing over all values of $n \geq k$ yields

$$\begin{aligned}\sum_{n=k}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} s'(t)s(t)^{k-1}\prod_{p=k+1}^{n} s(t-t_p)dt &= -\frac{1}{k} + \frac{1}{k}\sum_{n=k}^{\infty}\gamma n\left[S_n(t_{n+1}) - S_n(t_n)\right] \\ &\quad - \frac{1}{k}\sum_{p=k+1}^{\infty}\sum_{n=p}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} s(t)^k s'(t-t_p)\prod_{\substack{l=k+1\\ l\neq p}}^{n} s(t-t_l)dt\end{aligned} \tag{32}$$

where we have used the fact that $S_k(t_k) = 1$. Now, we note that the summand in the last term is exactly the second term in the gradient of $\nabla_p \mathcal{K}$ in equation (29). Summing equation (29) over $p > k$ yields

$$\sum_{p=k+1}^{\infty} \sum_{n=p}^{\infty} (1+n\gamma) \int_{t_n}^{t_{n+1}} s(t)^k s'(t-t_p) \prod_{\substack{l=k+1 \\ l\neq p}}^{n} s(t-t_l) dt$$
$$= -\sum_{p=k+1}^{\infty} \gamma S_p(t_p) + \sum_{p=k+1}^{\infty} \kappa \left(\sum_{n=1}^{p-1} \zeta(t_p - t_n) S_p(t_p) - \sum_{n=p+1}^{\infty} \zeta(t_n - t_p) S_n(t_n) \right) \tag{33}$$

Let us now rewrite the terms proportional to κ. First, let us note $A_{pn} = \zeta(t_p - t_n) S_p(t_p)$. Swapping the order of summation leads to

$$\begin{aligned}
\sum_{p=k+1}^{\infty} \left(\sum_{n=1}^{p-1} A_{pn} - \sum_{n=p+1}^{\infty} A_{np} \right) &= \sum_{p=k+1}^{\infty} \left(\sum_{n=1}^{k} A_{pn} + \sum_{n=k+1}^{p-1} A_{pn} - \sum_{n=p+1}^{\infty} A_{np} \right) \\
&= \sum_{p=k+1}^{\infty} \sum_{n=1}^{k} A_{pn} + \sum_{n=k+1}^{\infty} \sum_{p=n+1}^{\infty} A_{pn} - \sum_{p=k+1}^{\infty} \sum_{n=p+1}^{\infty} A_{np} \\
&= \sum_{p=k+1}^{\infty} \sum_{n=1}^{k} A_{pn} \\
&= \sum_{p=k+1}^{\infty} \sum_{n=1}^{k} s'(t_p) s(t_p)^{k-1} \prod_{l=k+1}^{p} s(t_p - t_l) \\
&= k \sum_{p=k+1}^{\infty} s'(t_p) s(t_p)^{k-1} \prod_{l=k+1}^{p} s(t_p - t_l)
\end{aligned} \tag{34}$$

where we have again used the fact that $t_n = 0$ for $n \leq k$ to evaluate A_{pn} in the sum. We therefore obtain

$$\sum_{p=k+1}^{\infty} \sum_{n=p}^{\infty} (1+n\gamma) \int_{t_n}^{t_{n+1}} s(t)^k s'(t-t_p) \prod_{\substack{l=k+1 \\ l\neq p}}^{n} s(t-t_l) dt$$
$$= -\sum_{p=k+1}^{\infty} \gamma S_p(t_p) + k\kappa \sum_{p=k+1}^{\infty} s'(t_p) s(t_p)^{k-1} \prod_{l=k+1}^{p} s(t_p - t_l) \tag{35}$$

Inserting this into equation (32) leads to

$$\sum_{n=k}^{\infty}(1+n\gamma)\int_{t_n}^{t_{n+1}} s'(t)s(t)^{k-1}\prod_{p=k+1}^{n} s(t-t_p)dt = -\frac{1}{k}+\frac{\gamma}{k}\sum_{n=k}^{\infty} n\left[S_n(t_{n+1})-S_n(t_n)\right]+\frac{\gamma}{k}\sum_{p=k+1}^{\infty} S_p(t_p) - \kappa\sum_{p=k+1}^{\infty} s'(t_p)s(t_p)^{k-1}\prod_{l=k+1}^{p} s(t_p-t_l) \quad (36)$$

We note that the last term of the latter equation compensates exactly the last term of $\nabla_k\mathcal{K}$ in equation (30). Bringing everything together yields

$$\nabla_k\mathcal{K} = -\gamma+\frac{1}{k}-\frac{\gamma}{k}\sum_{n=k}^{\infty} n\left[S_n(t_{n+1})-S_n(t_n)\right] - \frac{\gamma}{k}\sum_{n=k+1}^{\infty} S_n(t_n)+\kappa(k-1)s'(0) \quad (37)$$

Now, we rewrite

$$\begin{aligned}\sum_{n=k}^{\infty} n\left[S_n(t_{n+1})-S_n(t_n)\right] &= \sum_{n=k}^{\infty} nS_{n+1}(t_{n+1})-\sum_{n=k}^{\infty} nS_n(t_n)\\ &= \sum_{n=k+1}^{\infty}(n-1)S_n(t_n)-\sum_{n=k}^{\infty} nS_n(t_n)\\ &= -kS_k(t_k)-\sum_{n=k+1}^{\infty} S_n(t_n)\\ &= -k-\sum_{n=k+1}^{\infty} S_n(t_n) \quad (38)\end{aligned}$$

such that

$$-\frac{\gamma}{k}\sum_{n=k}^{\infty} n\left[S_n(t_{n+1})-S_n(t_n)\right] = \gamma+\frac{\gamma}{k}\sum_{n=k+1}^{\infty} S_n(t_n) \quad (39)$$

to finally obtain

$$\nabla_k\mathcal{K} = \frac{1}{k}+\kappa(k-1)s'(0) \quad (40)$$

N_{sim} is therefore found as the largest value of k which is such that this quantity is positive.